Experimental Notes on Elementary Differential Geometry

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Chapter 1

Introduction

The goal of these notes is to provide an introduction to differential geometry, first by studying geometric properties of curves and surfaces in Euclidean 3-space. Guided by what we learn there, we develop the modern abstract theory of differential geometry.

The approach taken here is radically different from previous approaches. Instead of working initially with $\mathbb{R}^n$ exclusively and introducing the modern abstract perspective later, I have chosen to introduce and use abstraction from the start. The philosophy here is to use abstraction whenever possible, and switch back to using coordinates on $\mathbb{R}^n$ only as needed.

Why do it this way, especially since abstraction can be difficult to understand? First, I believe that, although coordinates are at first sight easier to understand, they interfere with developing a clear conceptual understanding of the ideas. My view is that the abstract approach reflects more closely the geometric intuition driving the subject. Second, although coordinates are essential for proving some of the fundamental technical theorems, such as the implicit function theorem, most of the lemmas needed for laying the foundations of the subject are more easily proved using the abstract framework than using coordinates.

Last, the abstract perspective of mathematics is extremely useful, if not essential, for learning and doing advanced mathematics, even applied mathematics. Since it is so difficult to learn, it requires for most of us several attempts before we get it. The sooner you start struggling with it, the better.

Warning: All functions and maps are assumed to be continuous, C^1, or C^2 (we'll discuss what this means later), depending on the context. Many of the results presented do not hold if the functions or maps are assumed to be only differentiable.

Chapter 2

Euclidean geometry

Flat plane geometry was developed by Euclid using an axiomatic approach. He formulated a set of definitions and axioms (assumptions) and derived geometric theorems from them using only deductive logic. Later, Descartes observed that Euclidean plane can be represented using coordinates, and the theorems of Euclidean geometry proved using basic algebra. Some time after that, it was recognized that Descartes' approach can sometimes be done more elegantly using abstract algebra. Euclid's approach can also be extended to higher dimensional spaces.

But what is geometry? What is a geometric space, and when is a statement about the space called geometric? An initial answer comes from physics. When we study physics, we have to make measurements, which in turn require using units. For example, for length we can use inches, centimeters, or other units. However, we believe that the laws of physics should not depend on the units used. This means that a law of physics should hold, no matter what units are used. Ideally, one should be able to state the laws of physics without using any units at all. This can be done by writing the laws of physics in terms of unitless ratios.

In particular, in Newtonian physics it is assumed that empty space satisfies the geometric properties of Euclidean 3-space. Therefore, any law of physics has to be logically consistent with the axioms and theorems of Euclidean geometry. On the other hand, spatial measurements require units of distance and, in dimensions 2 or more, a way to measure angles. The most convenient way to do this is to use Cartesian coordinates. However, one must then check that any physical or geometric law remains valid and invariant under changes of coordinates.

Euclidean geometry can therefore be defined in one of two equivalent ways. It is the study of rigorous logical consequences of the Euclidean axioms. Or it is the study of theorems about Euclidean space, where the theorems and proofs might be stated using Cartesian coordinates but remain valid if the coordinates are changed by either shifting the origin to a different point in space or rotating the coordinate axes.

Today, Euclid's original axiomatic approach are rarely used, because the proofs are often quite subtle and difficult to find. On the other hand, although proofs using Cartesian coordinates are usually more straightforward, one must also verify that the theorem proved remains valid under a change of coordinates. The modern abstract algebraic approach allows theorems to be stated without any reference to coordinates, which often leads to short straightforward proofs, where the geometric invariance is immediately clear.

In this chapter we review briefly the three different approaches.

2.1 Axiomatic geometry

There are several different formulations of the axioms for Euclidean plane geometry. In all of them one starts with points, lines, and circles. Euclid himself first defined what are known as straightedge and compass constructions and then additional axioms. Others who formulated Euclidean geometry in terms of independent axioms include Hilbert, Birkhoff, and Tarski. Presented here are Hilbert's axioms.

2.1.1 Axioms of the line

We begin with axioms that specify the properties of a line and points on the line. Note that these axioms never refer to an origin, the length of a line segment, or numbers at all.

Order.

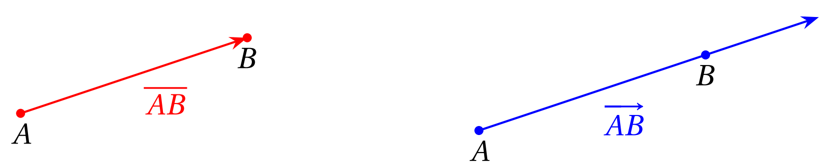
Concept of order. Given any three distinct points lying on a line, exactly one lies between the other two.

Existence of betweenness. Given two distinct points on a line, there exists a third distinct point on the line that lies between them.

Line segments and rays

Definition of oriented line segment. Given any two points A and B, the oriented line segment $\overline{AB}$ is defined to be the set containing A, B, and all points between A and B, along with the designation that A is the start and B is the end of $\overline{AB}$. In particular,

$$C \in \overline{AB} \iff C = A \text{ or } C = B \text{ or } C \text{ lies between } A \text{ and } B.$$



- *Definition of ray.* Given two distinct points A and B, the ray $\overrightarrow{AB}$ is defined to be the union of $\overline{AB}$ and the set of all points C such that B lies between A and the point in question. In particular,

$$C \in \overrightarrow{AB} \iff C = A \text{ or } C = B \text{ or } C \text{ lies between } A \text{ and } B \text{ or } B \text{ lies between } A \text{ and } C.$$

- *Directions of rays.* Two rays point in the same direction, if one contains the other.

- *Directions of oriented line segments.* Two oriented line segments $\overline{AB}$ and $\overline{CD}$ point in the same direction, if the rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ point in the same direction

Linear congruence.

Congruence of oriented line segments. There is an equivalence relation called *congruence* on the set of oriented line segments and denoted by $\cong$.

Existence of congruent line segments. Given a line segment $\overline{AB}$ and a ray $\overrightarrow{CD}$, there is a unique point E on the ray such that $\overline{AB} \cong \overline{CE}$.

Concatenation of line segments. Given line segments $\overline{AB}$ and $\overline{CD}$, let $\overline{AB} \vee \overline{CD}$ denote the line segment $\overline{AE}$, where $\overline{BE} \cong \overline{CD}$.

Multiple of a line segment. Given a line segment $\overline{AB}$, let $1\overline{AB} = \overline{AB}$. If $n \in \mathbb{Z}^+$, let

$$(n+1)\overline{AB} = n\overline{AB} \vee \overline{AB}.$$

Additivity of congruence. Given points A, B, C, D, E, F, G, H,

$$\overline{AB} \cong \overline{EF} \text{ and } \overline{CD} \cong \overline{GH} \implies \overline{AB} \vee \overline{CD} \cong \overline{EF} \vee \overline{GH}.$$

Topology of a line

Relative length. A line segment $\overline{AB}$ is *longer* than a line segment $\overline{CD}$, if D lies between C and E, where $\overline{CE}$ is the line segment in the ray $\overrightarrow{CD}$ that is congruent to $\overline{AB}$.

Archimedean axiom. For any line segments $\overline{AB}$ and $\overline{CD}$, there exists $n \in \mathbb{Z}^+$ such that $n\overline{AB}$ is longer than $\overline{CD}$.

Completeness. Let S, T be nonempty subsets of a line ℓ, such that $S \cup T = \ell$ and no point in one subset lies between two points in the other. Then there exists a unique point P such that for any points $A \in S\backslash\{P\}$ and $B \in T\backslash\{P\}$, P lies between A and B.

2.1.2 Axiomatic plane geometry

Incidence.

Unique line containing two points. For any two distinct points A and B, there exists a unique line $\overleftrightarrow{AB}$ passing though them.

Line has at least two points. Every line contains at least two distinct points.

Definition of collinearity. A set of points is collinear, if it is contained in a line.

Dimension is greater than 1. There exist three points that are not collinear.

Parallel axiom. For each line ℓ and point P not on ℓ, there exists a unique line through P that does not intersect ℓ.

Pasch's axiom. Suppose the points A, B, C are not collinear and ℓ is a line that does not pass through any of these points. If ℓ contains a point between A and B, then it also contains either a point between A and C or a point between B and C, but not both.

Angles.

Concept of angle. Any two distinct rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ defines an angle $\angle ABC$.

Concept of congruence. There is an equivalence relation called *congruence* on the set of angles and denoted $\cong$.

Existence of congruent angles. Given any angle $\angle ABC$ and a ray $\overrightarrow{ED}$, there exists a unique ray $\overrightarrow{EF}$ such that $\angle DEF \cong \angle ABC$.

Definition of triangle. A triangle consists of non-collinear points A, B, C and is denoted by $\triangle ABC$.

Congruence of triangles. Two triangles $\triangle ABC$ and $\triangle DEF$ are congruent, $\triangle ABC \cong \triangle DEF$, if

$$\overline{AB} \cong \overline{DE},\ \overline{BC} \cong \overline{EF},\ \overline{CA} \cong \overline{FD},$$

and

$$\angle ABC \cong \angle DEF,\ \angle BCA \cong \angle EFD,\ \angle CAB \cong \angle FDE.$$

SAS. If the triangles $\triangle ABC$ and $\triangle DEF$ satisfy

$$\overline{AB} \cong \overline{DE},\ \overline{BC} \cong \overline{EF}, \text{ and } \angle ABC \cong \angle DEF,$$

then they are congruent.

Circles

Definition of a circle. A circle centered at A is a set of points such that for any two points B and C lying in the circle, $\overline{AB} \cong \overline{AC}$.

Interior of a circle. A point C lies inside a circle centered at A, if there exists a point B in the circle such that C lies between A and B. A point lies outside a circle if it does not lie on or inside the circle.

Intersection of circles. If one circle contains both points inside and outside another circle, then the two circles intersect.

2.1.3 Consequences

Length and angles

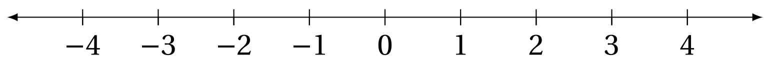
Areas of triangles and parallelograms

Pythagorean theorem

Arithmetic from geometry

2.2 Flat geometry using coordinates

2.2.1 Cartesian line



Given a line ℓ, choose a point $O \in \ell$, call it the origin, and label it by 0. Choose another point $E \neq O$ on the line and label it by 1. There exists a unique point $F \neq O$ such that $\overline{OE} \cong \overline{EF}$. Label F by 2. Continuing by induction, each nonnegative integer labels a unique point.

There exists a unique point $E' \neq E$ such that $\overline{E'O} \cong \overline{OE}$. Label it by -1. Continuing by induction, each integer labels a unique point.

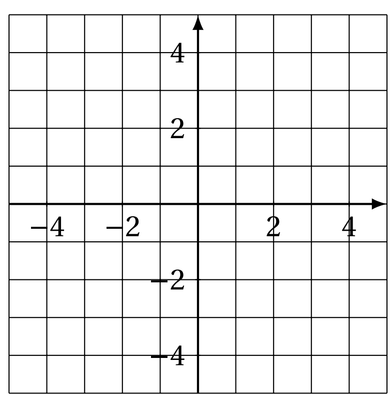
Each unit segment can be bisected. Each resulting smaller segment can also be bisected. Continuing by induction, each fraction whose denominator is a power of 2 labels a unique point.

By the completeness axiom, every real number labels a unique point, and every point is labeled by a unique real number. we call this the real line and denote it by $\mathbb{R}$.

Given any two points $a, b \in \mathbb{R}$, the line segment $\overline{ab}$ is congruent to a line segment $\overline{0c}$, where $c > 0$, if and only if $c = |b - a|$. Therefore,

$$\text{Length of } \overline{ab} = \text{ distance from } a \text{ to } b = |b - a|.$$

2.2.2 Cartesian plane



The Cartesian plane can be defined from the Euclidean plane as follows. Start with a point O on the plane. Choose a line passing through O and call it the x-axis. Choose a second line orthogonal to the x-axis and call it the y-axis. Choose a point $E_1 \neq O$ on the x-axis and a point

$E_2 \neq O$ on the y-axis such that $\overline{OE_1} \cong \overline{OE_2}$. By the construction of the Cartesian line, each axis is a real line, where E_1 and E_2 are labeled by 1.

For each point P in the plane, it follows by the parallel axiom that there exists a unique line passing through P and parallel to the y-axis. That line must intersect the x-axis. Let a denote the real number labeling the point of intersection. Similarly, there is a unique line through P that is parallel to the x-axis. Let b label the intersection of that line with the y-axis. Label P by the ordered pair (a,b). Let $\mathbb{R}^2$ denote the set of all possible ordered pairs (a,b), where $a,b \in \mathbb{R}$.

Conversely, given an ordered pair (a,b), there exist a unique line parallel to the y-axis and passing through a on the x-axis and a second unique line parallel to the x-axis and passing through b on the y-axis. These two lines must intersect, and the point of intersection is the point labeled by (a,b).

2.2.3 Length and distance

Given points $P_1 = (a_1,b_1)$ and $P_2 = (a_2,b_2)$, there exists a unique $c \geq 0$ such that $\overline{P_1P_2} \cong \overline{0c}$. We define the length of $\overline{P_1P_2}$ and the distance between P_1 and P_2 to be c. If we let $Q = (a_1,b_2)$, then $\triangle P_1QP_2$ is a right triangle. By the Pythagorean theorem,

$$(\ell(\overline{P_1P_2}))^2 = (\ell(\overline{P_1Q}))^2 + (\ell(\overline{QP_2}))^2.$$

This is equivalent to

$$(d(P_1,P_2))^2 = (a_2-a_1)^2 + (b_2-b_1)^2.$$

2.2.4 Angle via dot product

Recall that, if $P_0 = (a_0,b_0)$, $P_1 = (a_1,b_1)$, and $P_2 = (a_2,b_2)$, then

$$(P_1-P_0)\cdot(P_2-P_0) = (a_1-a_0)(a_2-a_0) + (b_1-b_0)(b_2-b_0).$$

By the law of cosines,

$$(P_1-P_0)\cdot(P_2-P_0) = d_1d_2\cos\theta,$$

, where

$$\begin{aligned} d_1 &= \sqrt{(a_1-a_0)^2+(b_1-b_0)^2} \\ d_2 &= \sqrt{(a_2-a_0)^2+(b_2-b_0)^2} \\ \theta &= \angle P_1P_0P_2. \end{aligned}$$

2.3 Abstract vector space with an inner product

Let $\mathbb{V}^m$ be an abstract m-dimensional vector space. A function

$$\begin{aligned} V \times V &\to \mathbb{R} \\ (v_1,v_2) &\mapsto \langle v_1,v_2\rangle \end{aligned}$$

is called an *inner product,* if the following hold for any $v, v_1, v_2 \in V$ and $c \in \mathbb{R}$:

$$\langle v_2, v_1\rangle = \langle v_1, v_2\rangle \text{ (symmetry)}$$

$$\left.\begin{array}{l}\langle v, v_1 + v_2\rangle = \langle v, v_1\rangle + \langle v, v_2\rangle \\ \langle v_1, cv_2\rangle = c\langle v_1, v_2\rangle\end{array}\right\} \text{ (bilinearity)}$$

$$\left.\begin{array}{r}\langle v, v\rangle \geq 0 \\ \langle v, v\rangle = 0 \iff v = 0\end{array}\right\} \text{ (positive definiteness).}$$

We can then define the length of a vector to be

$$|v| = \sqrt{\langle v, v\rangle}$$

and the angle θ between two nonzero vectors v_1 and v_2 by

$$\cos\theta = \frac{\langle v_1, v_2\rangle}{|v_1||v_2|}$$

The basic example of an abstract m-dimensional vector space is $\widehat{\mathbb{R}}^m$, and the basic example of an inner product is the dot product,

$$\langle v_1^1, \ldots, v_2^m\rangle \cdot \langle v_2^1, \ldots, v_2^m\rangle = v_1^1 v_2^1 + \cdots + v_1^m v_2^m$$

We will see later that this is essentially the only possible example. Nevertheless, we will also see later that it can be useful to forget about the Cartesian coordinates and use only the abstract definitions.

It is now straightforward to show that if you start with the Euclidean axioms, choose a point and call it the origin, then you can define vector addition and scalar multiplication on the Euclidean space, where the Euclidean axioms imply the properties listed above and the definitions of length and distance above are consistent with the definitions implied by the axioms.

Conversely, it is also straightforward to show that the properties of a vector space with an inner product imply the Euclidean axioms.

2.4 Euclidean geometry on an affine space

The vector space approach to Euclidean geometry requires choosing a special point in space and designating it as the origin. This can be avoided using affine geometry. In this approach, there is a space of points (which are not the same as vectors), called affine space, and an associated vector space, which we will call the tangent space, with an inner product. A detailed presentation is given in Chapter 3. It can be shown that the Euclidean axioms are equivalent to the definition of an affine space with an inner product on its tangent space.

The reason for introducing this perspective is that it provides a simple case of the more general concepts of a manifold and its tangent bundle.

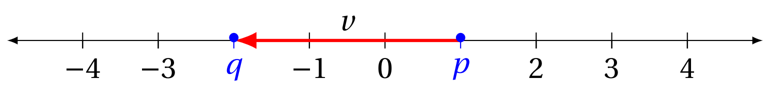
Chapter 3

Cartesian, Vector, Affine Spaces

3.1 1-dimensional spaces

3.1.1 The real line

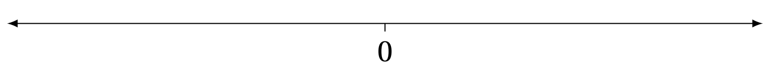
The real line is the 1-dimensional vector space but labeled by real numbers with respect to a chosen set of units.



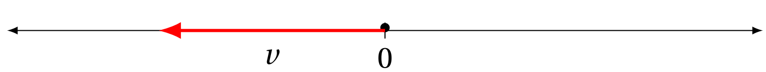
A real number represents either a point p on the number line or a displacement v from any point p to $p+v$. v can be viewed as a 1-dimensional vector and represented as an arrow, as shown. Given p and q on the real line, we can define the distance between p and q to be $|q-p| = |p-q|$.

3.1.2 Abstract 1-dimensional vector space

An abstract 1-dimensional vector space $\mathbb{V}^1$ can be thought of as the real line with all of the numbers, except 0, erased. It is, in particular, the set of all oriented line segments in a Euclidean line that start at zero. The concepts of congruence and rescaling of vectors are defined geometrically. They can in turn be used to define concepts of vector addition and rescaling of a vector that satisfy properties listed in Definition B.1.



A vector v looks like this in $\mathbb{V}^1$:



Without anything more, there is no way to measure the length of a vector. If, however, we declare a nonzero vector $e \in \mathbb{V}^1$ to be a unit vector, then, given any other vector v, there exists a unique real number t such that $v = te$. We can then define the length of v (relative to e) to be $|t|$. Once we do that, we can also define the distance between two vectors $v, w \in \mathbb{V}^1$ to be $|v-w|$.

3.1.3 The affine line

The affine line $\mathbb{A}^1$ can be viewed as an abstract line with the origin erased. It is the Euclidean line that satisfies the axioms in Subection 2.1.1. Associated with the affine line is the vector space $\mathbb{V}^1$.

Given two points $p, q \in \mathbb{A}$, there is a vector $v \in \mathbb{V}^1$ from p to q, which we can denote by $v = q - p$.

We denote $v = q - p \in \mathbb{V}^1$. Given a point p and a vector $v \in \mathbb{V}^1$, one can shift p by v, resulting in a point $q = p + v$, which is shown by the same picture.

Given any $p \in \mathbb{A}^1$, we can declare p to be the origin and make $\mathbb{A}^1$ look like the vector space $\mathbb{V}^1$. In particular, there is a natural bijection

$$\begin{aligned} I_p : \mathbb{V}^1 &\leftrightarrow \mathbb{A}^1 \\ v &\mapsto p + v \\ q - p &\mapsfrom q, \end{aligned}$$

where $I_p(0) = p$.

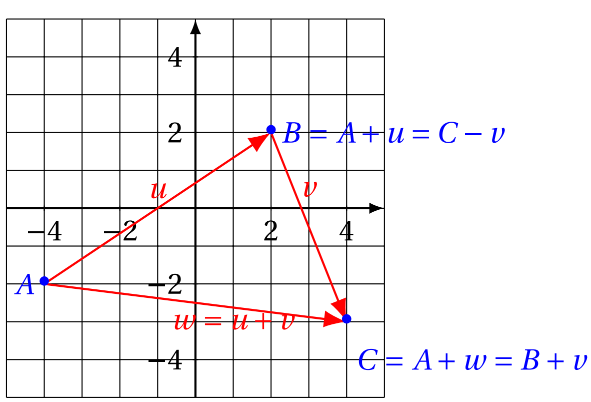
Given two different points $p, q \in \mathbb{A}^1$, we can declare p to be the origin and the distance from p to q to be 1. This leads to a natural bijection

$$\begin{aligned} J_p : \mathbb{R} &\to \mathbb{A}^1 \\ t &\mapsto p + tv, \end{aligned}$$

where $v = q - p \in V^1$ is assumed to have length 1.

3.2 2-dimensional spaces

3.2.1 Cartesian plane $\mathbb{R}^2$

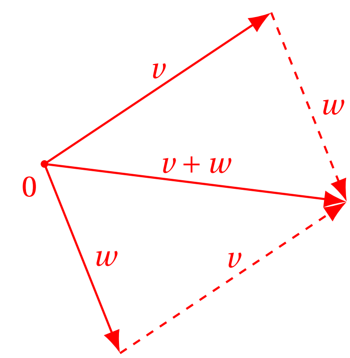


An element of $\mathbb{R}^2$ can be viewed as either a point or a vector. It is useful to distinguish between the two. We will denote a vector by $\langle v^1, v^2\rangle$ and the set of all such vectors by $\widehat{\mathbb{R}}^2$. A point will be denoted by (x^1, x^2) and the set of all points by $\widetilde{\mathbb{R}}^2$. In contrast to $\mathbb{R}^1$, a vector has both a length, defined using the Pythagorean theorem, and a direction. In this picture, we have the following relationships:

$$\begin{aligned} w &= u + v \\ v &= w - u \\ B - A &= u \\ C - B &= v \\ C - A &= w \\ A &= B - u = C - w \\ B &= A + u = C - v \\ C &= B + v = A + w. \end{aligned} \tag{3.1}$$

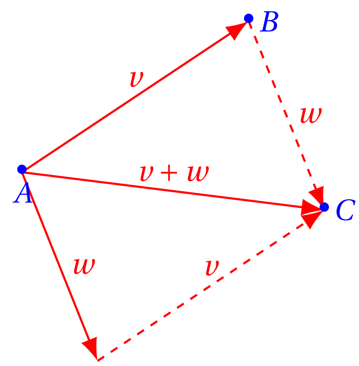
Observe that you can add or subtract two vectors and get a vector, subtract two points and get a vector, or add a vector to a point and get a point. You cannot, however, add two points.

3.2.2 2-dimensional abstract vector space $\mathbb{V}^2$



If you forget the Cartesian coordinates, but retain the geometric picture of $\mathbb{R}^2$ and the origin, you get a 2-dimensional abstract vector space. Observe that the equations in (3.1) still hold geometrically.

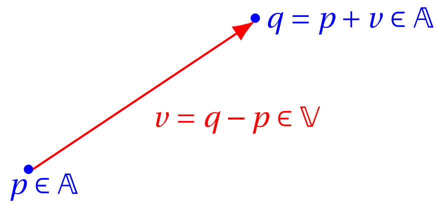
3.2.3 2-dimensional affine space



If you retain the picture of $\mathbb{V}^2$ but remove the origin, you get 2-dimensional affine space.

3.3 Abstract affine space

3.3.1 Definition



Definition 3.1. A set $\mathbb{A}$ is an *affine space with tangent space* $\mathbb{V}$, if the following hold:

1. $\mathbb{V}$ is an m-dimensional vector space
2. There is a point-point subtraction operator,
$$\begin{aligned}\mathbb{A} \times \mathbb{A} &\to \mathbb{V}\\ (p,q) &\mapsto q-p\end{aligned}$$
and a point-vector addition operator,
$$\begin{aligned}\mathbb{A} \times \mathbb{V} &\to \mathbb{A}\\ (p,v) &\mapsto p+v,\end{aligned}$$
which satisfy the following properties:
$$\begin{aligned}p+0 &= p\\ q-p=0 &\iff q=p\\ p+v=q &\iff v=q-p\\ p+(v_1+v_2) &= (p+v_1)+v_2.\end{aligned}$$

Given any $p \in \mathbb{A}$, the map
$$\begin{aligned}I_p : \mathbb{V} &\to \mathbb{A}\\ v &\mapsto p+v\end{aligned} \tag{3.2}$$
is bijective, and its inverse map is
$$\begin{aligned}I_p^{-1} : \mathbb{A} &\to \mathbb{V}\\ q &\mapsto q-p.\end{aligned}$$
For each $p, q \in \mathbb{A}$, the vector $I_p(q) = q-p$ is called the *position vector* of q relative to p.

Definition 3.2. The *dimension* of an affine space $\mathbb{A}$ is defined to be the dimension of the vector space associated to it.

3.3.2 Examples

1. Given any vector space $\mathbb{V}$, the space $\mathbb{A} = \mathbb{V}$ is itself an affine space with tangent space $\mathbb{V}$.

2. The hyperplane
$$H_1 = \{x^{m+1} = 1\} \subset \widehat{\mathbb{R}}^{m+1},$$
is an affine space with tangent space
$$H_0 = \{x^{m+1} = 0\} \subset \widehat{\mathbb{R}}^{m+1}.$$

3. More generally, given $a = \langle a_1, \ldots, a_{m+1} \rangle \in \widehat{\mathbb{R}}^{m+1} \backslash \{0\}$, the hyperplane
$$a^{-1}(1) = \{a_1 x^1 + \cdots a_{m+1} x^{m+1} = 1\} \subset \mathbb{R}^{m+1}.$$
is an affine space with tangent space
$$a^{\perp} = \{a_1 v^1 + \cdots a_{m+1} v^{m+1} = 0\} \subset \widehat{\mathbb{R}}^{m+1}.$$

4. The abstract version of the above is the following: Given an abstract vector space $\mathbb{V}$ and a nonzero linear function $\ell : \mathbb{V} \to \mathbb{R}$, the space
$$\ell^{-1}(1) = \{w \in \mathbb{V} \, : \, \ell(w) = 1\}$$
is an affine space with tangent space
$$\ell^{\perp} = \{v \in \mathbb{V} \, : \, \ell(v) = 0\}.$$
The space ℓ^{-1} is called an affine hyperplane of $\mathbb{V}$.

3.3.3 Affine independence

Recall that a set of vectors, $\{v_1, \ldots, v_k\} \subset \mathbb{V}$, is *linearly independent*, if the following holds: For any $a_1, \ldots, a_k \in \mathbb{R}$,
$$a^1 v_1 + \cdots a^k v_k = 0 \iff a^1 = \cdots = a^k = 0.$$

Definition 3.3. A set of points, $\{p_0, \ldots, p_k\} \subset \mathbb{A}$, is said to be or is *affinely independent* or in *general position*, if
$$\{p_1 - p_0, \ldots, p_k - p_0\} \subset \mathbb{V} \text{ is linearly independent.}$$

Lemma 3.4. *A set of points* $\{p_0, \ldots, p_k\} \subset \mathbb{A}$ *is in general position if and only if, for each* $0 \le i \le k$, *the set*
$$\{p_0 - p_i, \ldots, p_k - p_i\} \subset \mathbb{V}$$
consists of the zero vector and a linearly independent subset of $\mathbb{V}$.

3.3.4 Affine basis

Given a set of vectors, $S \subset \mathbb{V}$, define the *span* of S to be

$$[S] = \{a^1 v_1 + \cdots a^k v_k \;:\; a^1, \ldots, a^k \in \mathbb{R},\; v_1, \ldots, v_k \in \mathbb{V},\; k \geq 1\}.$$

Recall that an finite ordered list of vectors, $(v_1, \ldots, v_m)$ is a *linear basis* of a vector space $\mathbb{V}$, if the vectors are linearly independent and their span is all of $\mathbb{V}$. If a vector space $\mathbb{V}$ has a basis with m elements, we say that the dimension of $\mathbb{V}$ is m.

Recall that given such a basis, there is a bijective linear map

$$\begin{aligned} I_{(v_1,\ldots,v_m)} : \widehat{\mathbb{R}}^m &\to \mathbb{V} \\ \langle a^1, \ldots, a^m \rangle &\mapsto a^1 v_1 + \cdots a^m v_m. \end{aligned}$$

Definition 3.5. An ordered list of points, $(p_0, \ldots, p_m)$, is an *affine basis* of $\mathbb{A}$, if the vectors $(p_1 - p_0, \ldots, p_m - p_0)$ is a basis of $\mathbb{V}$.

Given an affine basis $P = (p_0, \ldots, p_n) \subset \mathbb{A}$, there is a bijective map

$$\begin{aligned} I_P : \widehat{\mathbb{R}}^m &\to \mathbb{A} \\ (a^1, \ldots, a^m) &\mapsto p_0 + a^1(p_1 - p_0) + \cdots + a^m(p_m - p_0). \end{aligned}$$

Observe that, if $E = (e_1, \ldots, e_m)$, where each $e_i = p_i - p_0$, then

$$I_P = I_{p_0}^{-1} \circ I_E,$$

where I_{p_0} is defined by (3.2) and I_E by (B.11).

3.3.5 Examples

1. If $(e_1, \ldots, e_{m+1})$ is the standard basis of $\mathbb{R}^{m+1}$, then $(e_{m+1}, e_{m+1} + e_1, \ldots, e_{m+1} + e_m)$ is an affine basis of H_1, as defined above.

2. Let $\ell, \ell^{-1}(1), \ell^{\perp}$ be as defined earlier. If the dimension of $\mathbb{V}$ is $m+1$, then for any basis $(e_1, \ldots, e_m)$ of $\ell^{\perp}$ and $e_0 \in \ell^{-1}(1)$, the ordered set $(e_0, e_1, \ldots, e_m)$ is an affine basis of $\ell^{-1}(1)$.

3.4 Affine maps

3.4.1 Definition of an affine map

Let $\mathbb{A}^m$ and $\mathbb{B}^n$ be affine spaces with corresponding tangent spaces $\mathbb{V}$ and $\mathbb{W}$. Given a map $M : \mathbb{A}^m \to \mathbb{B}^n$ and $p \in \mathbb{A}^m$, define the differential of M at at p to be the map

$$\begin{aligned} dM_p : \mathbb{V} &\to \mathbb{W} \\ v &\mapsto M(p+v) - M(p). \end{aligned}$$

Lemma 3.6. *For any $p, q \in \mathbb{A}^m$, dM_p is linear if and only if dM_q is linear.*

Definition 3.7. A map $A:\mathbb{A}^m \to \mathbb{B}^n$ is *affine*, if, for any $p \in \mathbb{A}^m$, the map $dM_p : \mathbb{V} \to \mathbb{W}$ is linear.

- A affine map is called a *affine isomorphism*, if it is both injective and surjective.
- The inverse of an affine isomorphism is also an affine isomorphism.
- An affine map $M:\mathbb{A}^m \to \mathbb{R}$ is called an *affine function*.
- The map $I_P : \mathbb{R}^m \to \mathbb{A}^m$ and its inverse are affine isomorphisms.

3.4.2 The space of affine maps

Let $\mathrm{Hom}_{\mathrm{aff}}(\mathbb{A}^m, \mathbb{B}^n)$ denote the space of all affine maps from $\mathbb{A}^m$ to $\mathbb{B}^n$.

Lemma 3.8. $\mathrm{Hom}_{\mathrm{aff}}(\mathbb{A}^m, \mathbb{B}^n)$ *is an affine space with tangent space* $\mathrm{Hom}_{\mathrm{aff}}(\mathbb{V}, \mathbb{W})$.

Observe that, given $p \in \mathbb{A}^m$, the map

$$\begin{aligned} I_p : \mathrm{Hom}(\mathbb{V}, \mathbb{W}) \times \mathbb{B}^n &\to \mathrm{Hom}_{\mathrm{aff}}(\mathbb{A}^m, \mathbb{B}^n) \\ (L, b) &\mapsto A, \end{aligned}$$

where, for each $q \in \mathbb{A}^m$,

$$A(q) = L(q-p) + b,$$

is an bijection. The inverse map $A \mapsto (L, b)$ is given by

$$\begin{aligned} L(v) &= A(p+v) - A(p) \\ b &= A(p) \end{aligned}$$

Given affine bases $P = (p_0, \dots, p_m)$ of $\mathbb{A}^m$ and $Q = (q_0, \dots, q_n)$ of $\mathbb{B}^n$, let

$$I_{P,Q} : \mathrm{Hom}_{\mathrm{aff}}(\mathbb{R}^m, \mathbb{R}^n) \to \mathrm{Hom}(\mathbb{A}^m, \mathbb{B}^n),$$

denote the map, where for each $M \in \mathrm{Hom}_{\mathrm{aff}}(\mathbb{R}^m, \mathbb{R}^n)$,

$$I_{P,Q}(M) = I_Q \circ M \circ I_P^{-1}.$$

Lemma 3.9. *The map $I_{P,Q}$ is an affine isomorphism.*

Lemma 3.10. $\dim \mathrm{Hom}(\mathbb{A}^m, \mathbb{B}^n) = (m+1)n$.

Chapter 4

Local Manifolds

4.1 Diffeomorphism

Recall that, if $O_1, O_2 \subset \mathbb{R}^m$ are open, then a map $\Phi : O_1 \to O_2$ is a C^k diffeomorphism if Φ is bijective and both Φ and Φ^{-1} are C^k maps.

4.2 C^k atlas

Given $m \geq 1$ and $k \geq 1$, a C^k atlas $\mathscr{A}$ of a set M is a nonempty set of bijections $\Phi : O \to M$, where $O \subset \mathbb{R}^m$ is open and, given any two such maps, $\Phi_1 : O_1 \to M$ and $\Phi_2 : O_2 \to M$, the map $\Phi_2^{-1} \circ \Phi_1 : O_1 \to O_2$ is a C^k diffeomorphism. The dimension of M is defined to be m.

We call any map $\Phi : O \to M$ in the atlas a *coordinate map* and the open set O a *coordinate chart*.

Given any such atlas $\mathscr{A}$, there is a maximal atlas $\mathscr{A}_{\max}$ consisting of all bijections $\widetilde{\Phi} : \widetilde{O} \to M$, where $\widetilde{O} \subset \mathbb{R}^m$ is open, such that, given any coordinate map $\Phi : O \to M$ in $\mathscr{A}$, the map $\Phi^{-1} \circ \widetilde{\Phi} : \widetilde{O} \to O$ is a C^k diffeomorphism. Any map in $\mathscr{A}_{\max}$ is also called a coordinate map and its domain a coordinate chart.

4.3 Local C^k manifold

Definition 4.1. A *local C^k manifold* is a set M with a C^k atlas $\mathscr{A}$.

Example. Any open set in $\mathbb{R}^m$ is a local C^k manifold, for any $k \geq 1$.

This appears to be an unnecessary definition, since M is essentially the same as an open subset of $\mathbb{R}^m$. However, the concept of a local manifold is analogous to the concept of an abstract vector space $\mathbb{V}^m$, where a coordinate map is analogous to a basis of $\mathbb{V}^m$ anda coordinate chart is analogous to $\mathbb{R}^m$.

This leads to a more abstract but less cluttered way to work with nonlinear spaces, analogous to how an abstract vector space can be easier to work with than $\mathbb{R}^m$. This is especially true, when we want to study properties of a local manifold that are independent of coordinates.

4.4 Maps between local manifolds

Given local C^k manifolds M^m and N^n, a map $F : M^m \to N^n$ is a C^k map, if, for any coordinate maps $\Phi : O^m \to M^m$ and $\widetilde{\Phi} : \widetilde{O}^n \to N^n$, the map $\widetilde{\Phi}^{-1} \circ F \circ \Phi : O^m \to \widetilde{O}^n$ is C^k.

4.5 Pointed local manifolds and maps

A pointed local manifold is a pair (M, p), where M is a local C^k manifold and $p \in M$. A map $F : (M, p) \to (N, q)$ is a C^k map $F : M \to N$ such that $F(p) = q$.

4.5.1 Coordinate map of a pointed local manifold

A coordinate map of a pointed local manifold (M, p) is a map $\Phi : (O, 0) \to (M, p)$, where $O \subset \mathbb{R}^m$ is an open neigborhood of the origin and $\Phi : O \to M$ is a coordinate map such that $\Phi(0) = p$.

4.6 Tangent space

Given a $p \in M$, we think of the tangent space at p, denoted T_pM or T_p to be the space of all possible velocity vectors of parameterized curves

4.6.1 Velocity vectors of curves

Given $p \in M$ and $\delta > 0$, let $\mathscr{C}_p$ be the set of all C^k parameterized curves $c : (-\delta, \delta) \to M$ such that $c(0) = p$. Given a coordinate map $\Phi : (O, 0) \to (M, p)$, and $c \in \mathscr{C}_p$, the curve $\hat{c} = \Phi^{-1} \circ c : (-\delta, \delta) \to O$ is C^k. Therefore, there is a map

$$\begin{aligned} \mathscr{C}_p &\to \mathbb{R}^m \\ c &\mapsto \hat{c}'(0). \end{aligned}$$

Define the following equivalence relation on $\mathscr{C}_p$:

$$c_1 \sim c_2 \iff (\Phi^{-1} \circ c_1)'(0) = (\Phi^{-1} \circ c_2)'(0)$$

Note that the definition of $\sim$ uses the coordinate map Φ. The crucial observation is that, despite this, $\sim$ in fact remains the same, no matter which coordinate map Φ is used. The precise statement is:

Lemma 4.2. *Let $\Phi_1 : (O_1, 0) \to (M, p)$ and $\Phi_2 : (O_2, 0) \to (M, p)$ be coordinate maps. For any $c_1, c_2 \in \mathscr{C}_p$, $c_1 \sim c_2$ with respect to a coordinate map Φ_1 if and only if $c_1 \sim c_2$ with respect to the coordinate map Φ_2.*

We can now define the tangent space at p to be the set of equivalence classes for $\sim$,

$$T_p = \mathscr{C}_p / \sim .$$

If we denote an equivalence class of $\sim$ by

$$[c] = \{\hat{c} \in \mathscr{C}_p \ : \ \hat{c} \sim c\},$$

then

$$T_p = \{[c] \ : \ c \in \mathscr{C}_p\}.$$

By the lemma, this definition of T_p does not depend on the coordinate map used.

Given any coordinate map $\Phi : (M, p) \to (O, 0)$, the definition of T_p implies that there is a bijective map

$$\Phi_* : T_p \to \mathbb{R}^m,$$

where $\Phi_*[c] = c'(0)$. Moreover, given any two coordinate maps Φ_1 and Φ_2,

$$(\Phi_2)_* \circ (\Phi_1)_* : \mathbb{R}^m \to \mathbb{R}^m$$

is a linear isomorphism. From this it follows that:

Lemma 4.3. *The tangent space T_p is a vector space, where, for each coordinate map $\Phi : M \to O$, the map $\Phi_* : T_p \to \mathbb{R}^m$ is a linear isomorphism.*

The set of tangent spaces for all $p \in M$,

$$T_* M = \cup_{p \in M} T_p,$$

is called the *tangent bundle* of M. An element of $T_* M$ can be written as (p, v), where $v \in T_p M$.

Lemma 4.4. *The tangent bundle of a local manifold M^m is a local manifold of dimension $2m$, whose coordinate maps include*

$$\begin{aligned} \widetilde{\Phi} : T_* M &\to O \times \mathbb{R}^m \\ (p, v) &\mapsto (\Phi(p), \Phi_* v), \end{aligned}$$

where $\Phi : (M, p) \to (O, 0)$ is a coordinate map.

4.6.2 Derivations

Here, we describe another equivalent definition of the tangent space. The idea here is that any vector defines a directional derivative of functions and there is a simple abstract characterization of a directional derivative.

Let $C^k(M)$ denote the space of C^k functions $f : M \to \mathbb{R}$. Given $p \in M$, a map

$$D_p : C^k(M) \to \mathbb{R}$$

is called a *derivation at p*, if it satisfies the following properties:

$$\begin{aligned} D(cf) &= cD(f) \\ D(f_1 + f_2) &= D(f_1) + D(f_2) \\ D(f_1 f_2) &= f_2(p) D(f_1) + f_1(p) D(f_2). \end{aligned}$$

Let T_pM denote the space of all derivations at p. Observe that it is a vector space.

Given any coordinate map $\Phi : (M,p) \to (O,0)$, there is a linear isomorphism

$$\Phi_* : T_pM \to T_0\mathbb{R}^m,$$

where, for each $D \in T_pM$, we define $\Phi_*(D)$ to be the derivation $\widetilde{D}$ given by

$$\widetilde{D}(\tilde{f}) = D(\tilde{f} \circ \Phi).$$

Lemma 4.5. *$\widetilde{D} \in T_0O$ if and only if there exists $v = (v^1, \dots, v^m) \in \mathbb{R}^m$ such that*

$$Df = v^1\partial_1 f(0) + \cdots v^m \partial_m f(0),$$

for any $f \in C^k(O)$.

4.7 Functions on a local manifold

Let M be a local C^k manifold. Given $0 \le j \le k$, a function $f : M \to \mathbb{R}$ is C^j if, given any coordinate map $\Phi : O \to M$, the function $f \circ \Phi : O \to \mathbb{R}$ is C^j.

4.7.1 Directional derivative

Given a tangent vector $v \in T_pM$, the directional derivative of f at p in the direction v is defined to be

$$d_v f(p) = \left.\frac{d}{dt}\right|_{t=0} f(c(t)),$$

where $c : (-\delta, \delta) \to M$ is a C^1 curve such that $c(0) = p$ and $c'(0) = v$.

4.7.2 Differential of function

Given a C^1 function $f : M \to \mathbb{R}$ and $p \in M$, the differential of f at p is defined to be the function

$$\begin{aligned} df(p) : T_p &\to \mathbb{R} \\ v &\mapsto d_v f(p). \end{aligned}$$

Since this is a linear function of $v \in T_p$, $df(p) \in T_p^*$.

4.8 Pushforward and pullback

4.8.1 Pushforward of a tangent vector by a map

Given C^k local manifolds M^m and N^n and a C^j map $F : M^m \to N^n$, where $1 \le j \le k$, the differential of F at p is the linear map

$$\begin{aligned} F_* : T_pM &\to T_{F(p)}N \\ v &\mapsto \left.\frac{d}{dt}(F \circ c)(t)\right|_{t=0}, \end{aligned} \tag{4.1}$$

where $c : (-\delta,\delta) \to M$ is a C^1 curve such that $c(0) = p$ and $c'(0) = v$ and therefore $F \circ c : (-\delta,\delta) \to N$ is a C^1 curve such that $F \circ c(0) = F(p)$. Given $v \in T_pM$, the tangent vector $F_* v \in T_{F(p)}N$ is called the *pushforward* of v by F.

4.8.2 Pullback of a cotangent vector by a map

Given $p \in M$, the dual map of F_* (see §B.9)

$$F^* : T^*_{F(p)}N \to T^*_p M,$$

is called the pullback map. Given $\theta \in T^*_{F(p)}N$, the covector $F^*\theta \in T^*_p M$ is called the *pullback* of θ by F.

The pushforward and pullback maps have the following property: Given any $v \in T_pM$ and $\theta \in T_{F(p)}N$,

$$\langle v, F^*\theta\rangle = \langle F_* v, \theta\rangle.$$

4.8.3 Pushforward and pullback in coordinates

Suppose M^m and N^n are local manifolds and $F : (M,p) \to (N,F(p))$ is a C^1 map. Let $\Phi : (O,0) \to (M,p)$ and $\widetilde{\Phi} : (\widetilde{O},0) \to (N,F(p))$ be coordinate maps. Consider the map

$$\begin{aligned} \widehat{F} = \widetilde{\Phi}^{-1} \circ F \circ \Phi &: (O,0) \to (\widetilde{O},0) \\ x = (x^1,\dots,x^m) &\mapsto \widehat{F}(x) = (y^1(x),\dots,y^n(x)), \end{aligned}$$

which is depicted by the commuting diagram

$$\begin{array}{ccc} (M,p) & \xrightarrow{F} & (N,F(p)) \\ {\scriptstyle\Phi}\uparrow & & {\scriptstyle\widetilde{\Phi}}\uparrow \\ (O,0) & \xrightarrow{\widehat{F}} & (\widetilde{O},0) \end{array}$$

The definitions of the tangent spaces T_pM, $T_{F(p)}N$ and the maps F_*, $\widehat{F}_*$, Φ_*, $\widetilde{\Phi}^*$ and the commuting diagram above imply the following commuting diagram:

$$\begin{array}{ccc} T_pM & \xrightarrow{F_*} & T_{F(p)}N \\ {\scriptstyle\Phi_*}\uparrow & & {\scriptstyle\widetilde{\Phi}_*}\uparrow \\ \mathbb{R}^m & \xrightarrow{\widehat{F}_*} & \mathbb{R}^n. \end{array}$$

In particular, if $v = v^i\partial_i \in T_pM$, then $\Phi_*^{-1} v = \langle v^1,\dots,v^m\rangle$. It follows that

$$\widehat{F}_*(0)v = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^m} \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix}$$

and

$$F_*(p)v = \left(\frac{\partial y^a}{\partial x^j} v^j\right)\frac{\partial}{\partial y^a}.$$

This in turn implies the following commuting diagram of dual linear maps:

$$\begin{array}{ccc} T_p^* M & \xleftarrow{F^*} & T_{F(p)}^* N \\ \downarrow \Phi^* & & \downarrow \widetilde{\Phi}^* \\ \mathbb{R}^m & \xleftarrow{\widehat{F}^*} & \mathbb{R}^n. \end{array}$$

In particular, if $\theta = \theta_a \, dy^a \in T_{F(p)} N$, then $\widetilde{\Phi}^*(0)\theta = \langle \theta_1, \ldots, \theta_n \rangle \in \mathbb{R}^n$. It follows that

$$\widehat{F}^* \theta = \begin{bmatrix} \theta_1 & \cdots & \theta_n \end{bmatrix} \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^m} \end{bmatrix}$$

and

$$F^* \theta = \theta_a \, dy^a = \theta_a (\partial_i y^a \, dx^i) = (\theta_a \partial_i y^a) \, dx^i$$

Chapter 5

A Little Bit of Category Theory

The concept of a category is implicit in much of the mathematics you have already learned. The idea is that, within a specific area such as linear algebra, you often restrict your attention to a certain class of sets with given properties, such as vector spaces, and a corresponding class of maps, such as linear transformations.

Another example is the class of open sets in $\mathbb{R}^n$ for any $n > 0$ and the corresponding class of smooth maps from an open set in $\mathbb{R}^n$ to $\mathbb{R}^m$ for any $m, n > 0$.

We will not use category theory explicitly. It is, however, useful simply as a way to keep track of how all the different mathematical objects and their properties fit together nicely.

5.1 Definition of a category

A category Cat consists of the following:

- A collection, denoted Obj, of things, usually called objects. For the categories considered here, objects are sets with certain specified properties, such vector spaces. We usually call an object a space.

- A collection, denoted Mor, of other things called morphisms or arrows. Associated with any morphism are two objects, called the source and target. Any morphism is written as $f : A \to B$, where f is the name of the morphism, A is the source, and B is the target. The notation is the same as for a map f with domain A and range B. For the categories considered here, a morphism is indeed a map from one space to another.

Moreover, the following properties must hold:

- (Composition) Any two morphisms $f : A \to B$, $g : B \to C$ (where the target of f is the source of g) uniquely determine a morphism with source A and target C, which we denote by $g \circ f : A \to C$.

- (Associativity) Given morphisms $f : A \to B$, $g : B \to C$, and $h : C \to D$,

$$(h \circ g) \circ f = h \circ (g \circ f) : A \to D.$$

  This property always holds when objects are sets and morphisms are maps.

- (Identity) For each object A, there is a morphism I_A such that, for any morphism $f : A \to B$, $f \circ I_A = f$ and, for any morphism $g : B \to A$, $I_A \circ g = g$.

Here, we will restrict to the following type of categories:

- Objects are sets that satisfy a common set of properties. Such an object is often called a space.
- A morphism is a map between two spaces that satisfies a specified set of properties.
- The map $g \circ f$ is the composition of the two maps g and f.
- $I_A : A \to A$ is the identity map.

5.2 Examples

5.2.1 Sets

Objects

Sets

Morphisms

Maps from a set to another

5.2.2 Finite dimensional real vector spaces

Objects

Finite dimensional real vector spaces

Morphisms

Linear maps

5.2.3 Finite dimensional affine spaces

Objects

Finite dimensional real affine spaces

Morphisms

Affine maps

5.2.4 C^k Cartesian manifolds

Objects

Open subsets of $\mathbb{R}^m$, for any m

Morphisms

C^k maps

5.3 Definition of Functor

A *covariant functor* $\mathscr{F} : \mathrm{Cat}_1 \to \mathrm{Cat}_2$ consists of maps

$$\mathscr{F} : \mathrm{Obj}_1 \to \mathrm{Obj}_2$$
$$\mathscr{F} : \mathrm{Mor}_1 \to \mathrm{Mor}_2,$$

such that the following hold:

- For each map $f : X \to Y$ in Mor_1, the domain of $\mathscr{F}(f)$ is $\mathscr{F}(X)$ and its range is $\mathscr{F}(Y)$.
- Given maps $f : X \to Y$ and $g : Y \to Z$,
$$\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f)$$
- For each space X,
$$\mathscr{F}(\mathrm{id}_X) = \mathrm{id}_{\mathscr{F}}(X).$$

A *contravariant functor* $\mathscr{F} : \mathrm{Cat}_1 \to \mathrm{Cat}_2$ reverses the arrows. It consists of maps

$$\mathscr{F} : \mathrm{Obj}_1 \to \mathrm{Obj}_2$$
$$\mathscr{F} : \mathrm{Mor}_1 \to \mathrm{Mor}_2,$$

such that the following hold:

- For each map $f : X \to Y$ in Mor_1, the domain of $\mathscr{F}(f)$ is $\mathscr{F}(Y)$ and its range is $\mathscr{F}(X)$.
- Given maps $f : X \to Y$ and $g : Y \to Z$,
$$\mathscr{F}(g \circ f) = \mathscr{F}(f) \circ \mathscr{F}(g)$$
- For each space X,
$$\mathscr{F}(\mathrm{id}_X) = \mathrm{id}_{\mathscr{F}(X)}.$$

5.4 Examples of functors

5.4.1 Forgetful functors

A category is a subcategory of another category, if any object of the subcategory is an object of the category and any morphism of the subcategory is a morphism of the category. The forgetful functor from the subcategory to the category is simply the one where $\mathscr{F}(X) = X$ for any object X and $\mathscr{F}(f) = f$ for any map f.

There is a forgetful functor $\mathscr{F}$ from each category above to the category of sets, where $\mathscr{F}(X) = X$ and $\mathscr{F}(f) = f$.

There is a forgetful functor from the category of vector spaces to the category of affine spaces.

There is a forgetful functor from the category of affine spaces to the category C^{k+1} Cartesian manifolds.

There is a forgetful functor from the category of C^{k+1} Cartesian manifolds to the category of C^k Cartesian manifolds.

5.4.2 Dualization

There is a contravariant functor *Dual* from the category of vector spaces to itself, where, for each vector space V, $Dual(V) = V^*$ and, for each linear map $L: V \to W$, $Dual(L) = L^t : W^* \to V^*$.

5.5 Categories of linear, affine, and nonlinear spaces

5.5.1 Categories

Category	**Object**	**Morphism**
Cartesian spaces	$\mathbb{R}^m$	Linear map $L: \mathbb{R}^m \to \mathbb{R}^n$
Vector spaces	Vector space $\mathbb{V}^m$	Linear map $L: \mathbb{V}^m \to \mathbb{W}^n$
Affine spaces	Affine space $\mathbb{A}^m$	Affine map $A: \mathbb{A}^m \to \mathbb{B}^n$
Cartesian manifolds	Open set $O^m \subset \mathbb{R}^m$	C^k map $f: O^m \to \widetilde{O}^n$
Local manifolds	Open set $O^m \subset \mathbb{A}^m$	C^k map $f: O^m \to \widetilde{O}^n$
C^k Manifolds	C^k Manifold M^m	C^k map $f: M^m \to N^n$
Pointed affine spaces	$(\mathbb{A}^m, p)$	$A: (\mathbb{A}^m, p) \to (\mathbb{B}^n, q)$, $A(p) = q$
Pointed Cartesian manifolds	(O^m, p)	C^k map $f: (O^m, p) \to (\widetilde{O}^n, q)$, $f(p) = q$
Pointed C^k Manifolds	(M^m, p)	$f: (M^m, p) \to (N^n, q)$, $f(p) = q$

5.5.2 Linearization functor

From Category	To Category	From Oject	To Object	From Morphism	To Morphism
Pointed affine spaces	Vector spaces	$(\mathbb{A}^m, p)$	$\mathbb{V}^m = \mathbb{A}^m - p$	$A : (\mathbb{A}^m, p) \to (\mathbb{B}^n, q)$	$L : \mathbb{V}^m \to \mathbb{W}^m$ $L(v) = A(p+v) - q$
Pointed Cartesian manifolds	Cartesian spaces	(O^m, p)	$\mathbb{R}^m$	$f : (O^m, p) \to (\widetilde{O}^n, q)$	$\partial f(p) : \mathbb{R}^m \to \mathbb{R}^n$
Pointed C^k manifolds	Vector spaces	(M, p)	T_pM	$f : (M, p) \to (N, q)$	$f_* : T_pM \to T_qN$
Pointed C^k manifolds	Vector spaces	(M, p)	T_p^*M	$f : (M, p) \to (N, q)$	$f^* : T_q^*N \to T_pM$

5.5.3 Pushforward and pullback functors

The linearization functor is also called the pushforward functor. Composing with the dualization functor for vector spaces yields the pullback functor.

Pushforward

$$\begin{array}{ccc} (\mathbb{A}^m, p) & \xrightarrow{A} & (\mathbb{B}^n, q) \\ \downarrow & & \downarrow \\ \mathbb{V}^m & \xrightarrow{L} & \mathbb{W}^n \end{array}$$

$$\begin{array}{ccc} (O^m, p) & \xrightarrow{f} & (\widetilde{O}^n, q) \\ \downarrow & & \downarrow \\ \mathbb{R}^m & \xrightarrow{f_*} & \mathbb{R}^n \end{array}$$

$$\begin{array}{ccc} (M, p) & \xrightarrow{f} & (N, q) \\ \downarrow & & \downarrow \\ T_pM & \xrightarrow{f_*} & T_qN \end{array}$$

Pullback

$$\begin{array}{ccc} (\mathbb{A}^m, p) & \xrightarrow{A} & (\mathbb{B}^n, q) \\ \downarrow & & \downarrow \\ \mathbb{V}^* & \xleftarrow{L^t} & \mathbb{W}^* \end{array}$$

$$\begin{array}{ccc} (O^m, p) & \xrightarrow{f} & (\widetilde{O}^n, q) \\ \downarrow & & \downarrow \\ \mathbb{R}^m & \xleftarrow{(\partial f(p))^t} & \mathbb{R}^n \end{array}$$

$$\begin{array}{ccc} (M, p) & \xrightarrow{f} & (N, q) \\ \downarrow & & \downarrow \\ T_p^*M & \xleftarrow{f^*} & T_q^*N \end{array}$$

Chapter 6

Euclidean space

The concepts of a vector space and an affine space provide an abstract algebraic formulation of the properties of points and lines in Euclidean space. In this chapter we introduce an abstract version of the dot product, which is then used to define the length of a vector and the angle between two vectors in an abstract vector space. This in turn can be used to define the distance between two points in an affine space and the angle defined by three points.

6.1 Cartesian space

6.1.1 Length and angle using the dot product

Recall that the dot product of two vectors in $\widehat{\mathbb{R}}^m$ is defined to be

$$\langle v_1^1, \dots, v_1^m \rangle \cdot \langle v_2^1, \dots, v_2^m \rangle = v_1^1 v_2^1 + \cdots + v_1^m v_2^m. \tag{6.1}$$

The *length* or *magnitude* of a vector is

$$|v| = \sqrt{v \cdot v}.$$

Recall that, if the angle between two nonzero vectors v_1 and v_2 is θ, where $0 \le \theta \le \pi$, then

$$\cos\theta = \frac{v_1 \cdot v_2}{|v_1||v_2|}.$$

In particular, the two vectors are orthogonal, if

$$v_1 \cdot v_2 = 0,$$

and, if this holds,

$$|v_1|^2 + |v_2|^2 = |v_2 - v_1|^2,$$

which is the Pythagorean theorem.

The dot product also has the following fundamental properties: For each $v_1, v_2, v_3 \in \widehat{\mathbb{R}}^m$ and $a \in \mathbb{R}$,

- Multilinearity:

$$\begin{aligned}(v_1 + v_2)\cdot v_3 &= v_1\cdot v_3 + v_2\cdot v_3\\ (av_1)\cdot v_2 &= a(v_1\cdot v_2)\\ v_1\cdot(v_2+v_3) &= v_1\cdot v_2 + v_1\cdot v_3\\ v_1\cdot(av_2) &= a(v_1\cdot v_2).\end{aligned} \tag{6.2}$$

- Symmetry:

$$v_1\cdot v_2 = v_2\cdot v_1. \tag{6.3}$$

- Positive definiteness:

$$\begin{aligned}&v_1\cdot v_1 \geq 0\\ &v_1\cdot v_1 = 0 \iff v_1 = 0.\end{aligned} \tag{6.4}$$

This can be summarized by the fact that the dot product is a positive definite symmetric 2-tensor.

6.2 Inner product

It turns out that almost all geometric theorems about Euclidean space can be proved using only the properties of the dot product, and its definition (6.1) is rarely needed. Due to this, it turns out to be useful to define an abstract concept of the dot product, which is usually called the inner product.

Definition 6.1. An *inner product* on a vector space V is a positive definite symmetric 2-tensor, which will be denoted just like the dot product,

$$v_1, v_2 \in \mathbb{V} \mapsto v_1\cdot v_2 \in \mathbb{R}.$$

In particular, it satisfies the properties (6.2), (6.3), and (6.4).

Given an inner product on $\mathbb{V}$, we define the *norm* or *magnitude* of a vector $v\in\mathbb{V}$ to be

$$|v| = \sqrt{v\cdot v}. \tag{6.5}$$

An *inner product space* is a vector space with an inner product. Cartesian space $\widetilde{\mathbb{R}}^m$ with the dot product is an inner product space. It is, however, often useful to work with an abstract inner product space.

Lemma 6.2. *Given a basis $(e_1,\dots,e_m)$ of a vector space $\mathbb{V}^m$, an inner product on $\mathbb{V}^m$ is uniquely determined by the symmetric matrix*

$$A = \begin{bmatrix} A_{11} & \dots & A_{1m}\\ \vdots & \dots & \vdots\\ A_{m1} & \dots & A_{mm}\end{bmatrix},$$

where $A_{ij} = e_i \cdot e_j$. *Bilinearity implies that, given vectors* $v = v^i e_i$ *and* $w = w^i e_i$,

$$v \cdot w = \begin{bmatrix} v^1 & \cdots & v^m \end{bmatrix} A \begin{bmatrix} w^1 \\ \vdots \\ w^m \end{bmatrix}. \tag{6.6}$$

On the other hand, a symmetric matrix A, (6.6) does not necessarily define an inner product. Simple counterexamples for $m = 2$ include

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Definition 6.3. A symmetric m-by-m matrix A is *positive definite*, if (6.6) defines an inner product.

Lemma 6.4. *A symmetric* 2*-by-*2 *matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{12} = a_{21}$, *defines an inner product if and only if*

$$a_{11}, a_{22} > 0 \textit{ and } a_{11} a_{22} - a_{12}^2 > 0.$$

6.3 Orthonormal vectors

Let $\mathbb{V}^m$ be m-dimensional inner product space.

Definition 6.5. A set of vectors,

$$E = \{e_1, \dots, e_k\} \subset \mathbb{V}^m$$

is *orthonormal*, if for any $1 \le i, j \le k$,

$$e_i \cdot e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}.$$

Note that if $v \in [E]$, then

$$v = \sum_{i=1}^{k} a^i e_i,$$

where, for each $1 \le i \le k$,

$$a^i = v \cdot e_i.$$

Lemma 6.6. *A set of orthonormal vectors is linearly independent.*

For each orthonormal set E, define

$$[E]^{\perp} = \{v \in \mathbb{V}^m \ : \ v \cdot e = 0, \ \forall e \in [E]\}$$

and the linear maps

$$\begin{aligned}\pi_E : \mathbb{V}^m &\to [E] \\ v &\mapsto \sum_{i=1}^{i=k} (v \cdot e_i) e_i \\ \pi_E^{\perp} : \mathbb{V}^m &\to \mathbb{V}^m \\ v &\mapsto v - \pi_E(v).\end{aligned}$$

Lemma 6.7. *For each $v \in \mathbb{V}^m$,*

$$v = v_1 + v_2, \text{ where } v_1 \in [E] \text{ and } v_2 \in [E]^{\perp}$$

if and only if

$$v_1 = \pi_E(v) \text{ and } v_2 = \pi_E^{\perp}(v).$$

Lemma 6.8. *A finite-dimensional inner product space has at least one orthornomal basis.*

Lemma 6.9. *If $E = (e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{V}^m$, then $F = (f_1, \ldots, f_m)$ is also an orthonormal basis if and only if*

$$f_j = e_i M^i_j, \ \forall \ 1 \le j \le m,$$

where

$$M^i_j M^i_k = \delta_{jk}, \ \forall \ 1 \le j, k \le m,$$

or, equivalently, if and only if

$$F = EM,$$

where

$$M^t M = I. \tag{6.7}$$

6.4 Orthogonal maps

Definition 6.10. An *orthogonal map* is a map $F : \mathbb{V}^m \to \mathbb{W}^n$, where $\mathbb{V}^m$ and $\mathbb{W}^n$ are inner product spaces, such that

$$F(v_1) \cdot F(v_2) = v_1 \cdot v_2, \ \forall \ v_1, v_2 \in \mathbb{V}^m.$$

If $\mathbb{W}^n = \mathbb{V}^m$, then F is also called an orthogonal transformation.

Lemma 6.11. *An orthogonal map is linear.*

Let $\mathrm{O}(\mathbb{V}^m, \mathbb{W}^n)$ denote the set of all orthogonal maps from $\mathbb{V}^m$ to $\mathbb{W}^n$ and $\mathrm{O}(\mathbb{V}^m) = \mathrm{O}(\mathbb{V}^m, \mathbb{V}^m)$. The latter is a group.

Let $\mathrm{O}(m)$ denote the set of all m-by-m matrices satisfying (6.7). It is a group under matrix multiplication.

Lemma 6.12. *Given an orthonormal basis E of an inner product space $\mathbb{V}^m$, there is a group homomorphism*

$$\begin{aligned} \mathrm{O}(m) &\to \mathrm{O}(\mathbb{V}^m) \\ M &\mapsto A, \end{aligned}$$

where, for each $v = e_i a^i$,

$$A(e_i a^i) = e_j M^j_i a^i.$$

6.5 Metric spaces

Recall that the distance between two points $p_1, p_2 \in \widetilde{\mathbb{R}}^m$ is given by

$$d(p_1, p_2) = |p_2 - p_1| = \sqrt{(p_2 - p_1) \cdot (p_2 - p_1)}.$$

and satisfies the following properties for any $p_1, p_2, p_3 \in \widetilde{\mathbb{R}}^m$:

$$\begin{aligned} &d(p_1, p_2) = d(p_2, p_1) \\ &d(p_1, p_2) \geq 0 \\ &d(p_1, p_2) = 0 \iff p_2 = p_1 \\ &d(p_1, p_3) \leq d(p_1, p_2) + d(p_2, p_3). \end{aligned} \tag{6.8}$$

The leads to the following abstract definition:

Definition 6.13. A *metric space* is a set X and a function $d : X \times X \to [0, \infty)$ that satisfies, for every $p_1, p_2, p_3 \in X$, the properties given by (6.8).

If X and Y are both metric spaces, then a map $F : X \to Y$ is called an *isometry*, if for any $x_1, x_2 \in X$,

$$d_Y(F(x_1), F(x_2)) = d_X(x_1, x_2).$$

If $\mathbb{A}$ is an affine space with tangent space $\mathbb{V}$ and $|\cdot|$ is a norm on $\mathbb{V}$, then $\mathbb{A}$ with the distance function

$$d(p_1, p_2) = |p_2 - p_1|$$

is a metric space.

6.6 Isometries of Euclidean space

Definition 6.14. An m-dimensional Euclidean space $\mathbb{E}^m$ is an m-dimensional affine space whose tangent space is an inner product space. Since the inner product defines a norm on the tangent space, $\mathbb{E}^m$ is a metric space.

Lemma 6.15. *Let $\mathbb{A}^m$ and $\mathbb{B}^n$ be Euclidean spaces. A map $F : \mathbb{A}^m \to \mathbb{B}^n$ is an isometry, if and only if there exists $L \in \mathrm{O}(\mathbb{V}^m, \mathbb{W}^n)$ such that the following holds: For any $p \in \mathbb{A}^m$, there exists $q \in \mathbb{B}^n$ such that, for any $a \in \mathbb{A}^m$,*

$$F(a) = L(a - p) + q.$$

Definition 6.16. An isometry $F : \mathbb{A}^m \to \mathbb{A}^m$ is called a *rigid motion*.

Lemma 6.17. *The set G of all possible rigid motions of m-dimensional Euclidean space is a group, because it satisfies the following properties:*

1. *If $F_1, F_2 \in G$, then $F_2 \circ F_1 \in G$.*
2. *The identity map $I : \mathbb{A}^m \to \mathbb{A}^m$ is in G.*
3. *Any rigid motion $F : \mathbb{A}^m \to \mathbb{A}^m$ has an inverse map $F^{-1} : \mathbb{A}^m \to \mathbb{A}^m$ that is also a rigid motion.*

Chapter 7

Curves

7.1 Curves in $\mathbb{R}^m$

Definition 7.1. A parameterized C^k curve in $\mathbb{R}^m$ is a map of the form

$$\begin{aligned} c : I &\to \mathbb{R}^m \\ t &\mapsto c(t) = (c^1(t), \ldots, c^m(t)), \end{aligned}$$

where $I \subset \mathbb{R}$ is an interval and each c^i is a C^k function on I.

The *velocity* of a C^1 curve $c : I \to \mathbb{R}^m$ is defined to be $\dot{c} : I \to \mathbb{R}^m$, where

$$\dot{c}(t) = \langle \dot{c}^1(t), \ldots, \dot{c}^m(t) \rangle$$

and, for each $1 \le i \le m$,

$$\dot{c}^i = \frac{dc^i}{dt}.$$

The *acceleration* of a C^2 curve c is defined to be $\ddot{c} : I \to \mathbb{R}^m$, where

$$\ddot{c} = \frac{d\dot{c}}{dt} = \frac{d^2 c}{dt^2}.$$

7.2 Curves in affine space

A parameterized C^k curve in $\mathbb{A}^m$ is a map of the form

$$c : I \to \mathbb{A}^m,$$

where $I \subset \mathbb{R}$ is an interval and, given any affine basis $P = (p_0, \ldots, p_m)$ of $\mathbb{A}^m$, $I_P^{-1} \circ c : I \to \mathbb{R}^m$ is a C^k curve as defined above.

Definition 7.2. The velocity of a C^1 curve $c : I \to \mathbb{A}^m$ at time $t_0 \in I$ is defined to be the derivative of c at time t_0,

$$v(t) = \dot{c}(t) = \lim_{s \to t} \frac{c(s) - c(t)}{s - t} \in \mathbb{V}^m.$$

Since $c'(t) \in \mathbb{V}^m$ is tangent to $c(t) \in \mathbb{A}^m$, the vector space $\mathbb{V}^m$ is called the *tangent space* of $\mathbb{A}^m$.

If c is C^2, then the acceleration is the second derivative of c,

$$a(t) = \ddot{c}(t) = \lim_{s \to t} \frac{c'(s) - c'(t)}{s - t} \in \mathbb{V}^m.$$

7.3 Curves in Euclidean space

Let $\mathbb{E}^m$ be m-dimensional Euclidean space with tangent space $\dot{\mathbb{E}}^m$, which has an inner product.

7.3.1 Velocity, speed, and length

Given a C^1 curve $c : [t_0, t_1] \to \mathbb{E}^m$ with velocity $v : I \to \dot{\mathbb{E}}^m$, its *speed* is defined to be

$$\sigma = |v| : I \to [0, \infty).$$

The distance along the curve, i.e., the length of the curve, between two points $c(t_0)$ and $c(t_1)$ is therefore

$$\ell = \int_{t=t_0}^{t=t_1} \sigma(t)\, dt.$$

In particular, we can define the arclength function to be

$$\begin{aligned} [t_0, t_1] &\to [0, \ell] \\ t &\mapsto s(t) = \int_{\tau=t_0}^{\tau=t} \sigma(\tau)\, d\tau. \end{aligned}$$

If the velocity is never zero, then σ is continuous and by the fundamental theorem of calculus, $s' = \sigma$. Moreover, s is strictly increasing, and therefore, by the chain rule, has a C^1 inverse function $t(s)$. The *arclength parameterization* of c is defined to be

$$\begin{aligned} \hat{c} : [0, \ell] &\to \mathbb{E}^m \\ s &\mapsto c(t(s)). \end{aligned}$$

Since the speed of $\hat{c}$ is 1, it is called a *unit speed curve*.

A simple fact we will use is the following: Given any C^1 function $f : [t_0, t_1] \to \mathbb{R}$,

$$\frac{d}{ds}(f(t(s))) = \frac{f'(t(s))}{s'(t)}.$$

7.3.2 Acceleration and curvature

The velocity of a C^2 curve $c : I \to \mathbb{E}^m$ can be written as

$$v(t) = \sigma(t) u(t), \text{ where } u = \frac{v}{|v|} \text{ is a unit vector for each } t \in I.$$

Therefore, the acceleration of c is the sum of two components:

$$a = \dot{v} = \dot{\sigma} u + \sigma \dot{u}.$$

Since $|u| = 1$,

$$0 = \frac{d}{dt}|u|^2 = \frac{d}{dt}(u \cdot u) = 2u \cdot \dot{u}. \tag{7.1}$$

It follows that the two components of the acceleration a are orthogonal. The first term represents the acceleration of speed along the curve, while the second represents the acceleration due to the change in direction of the curve.

7.3.3 Curvature

Geometric properties of the curve should not depend on the parameterization. We therefore want properties that do not depend on the speed or acceleration along the curve. It suffices to study only unit speed curves. Notice that this restricts us to curves that have at least one parameterization with nonzero velocity everywhere along the curve.

The other thing to note is that a curve is straight if u, as defined above, never changes direction. Moreover, the greater the magnitude of u' is, the more sharply curved the curve is. It therefore makes sense to define the curvature at each point $c(s)$ of a **unit speed** C^2 curve to be

$$\kappa(s) = |a(s)| = |u'(s)|.$$

Observe that

$$\begin{aligned} c'(s) &= u(s) \\ |c''(s)| &= |u'(s)| \\ &= \kappa(s) \end{aligned}$$

7.3.4 Frenet-Serret frame in Euclidean 2-space

Let $c : [0, T] \to \mathbb{E}^2$ be a unit speed C^2 curve. Let $e_1 = \dot{c}$ There exists, for each $t \in [0, T]$, a unit vector $e_2(t)$

$$E(t) = \begin{bmatrix} e_1(t) & e_2(t) \end{bmatrix} \tag{7.2}$$

is an orthonormal basis of $\dot{\mathbb{E}}^2$. By (7.1), $e_1 \cdot e_1' = 0$. This implies that $e_1'(t)$ is a scalar multiple of e_2. It follows that there is a function κ such that, for each $t \in [0, T]$,

$$e_1'(t) = \sigma(t)\kappa(t)e_2(t).$$

Again, since e_2' is orthogonal to e_2, it is a scalar multiple of e_1. On other hand, since $e_1 \cdot e_2 = 0$,

$$0 = \frac{d}{dt}(e_1 \cdot e_2) = e_1' \cdot e_2 + e_1 \cdot e_2' = \sigma\kappa + e_1 \cdot e_2',$$

and therefore

$$e_1 \cdot e_2' = -\sigma\kappa.$$

It follows that

$$e_2' = -\sigma\kappa e_1.$$

The 1-parameter family $E(t)$ of orthonormal bases of $\dot{\mathbb{E}}^2$ is called the *Frenet-Serret frame* of c. We have shown that it satisfies the equations

$$\begin{aligned}\begin{bmatrix} e_1'(t) & e_2'(t)\end{bmatrix} &= \sigma \begin{bmatrix}\kappa e_2(t) & -\kappa e_1(t)\end{bmatrix} \\ &= \sigma \begin{bmatrix} e_1(t) & e_2(t)\end{bmatrix} \begin{bmatrix} 0 & -\kappa(t) \\ \kappa(t) & 0\end{bmatrix}.\end{aligned} \tag{7.3}$$

This is known as the Frenet-Serret equations in Euclidean 2-space.

If we however, assume that $c : [0,T] \to \mathbb{E}^2$ is a C^2 unit speed curve such that a Frenet-Serret frame satisfying the Frenet-Serret equations(7.3), then the curvature function uniquely determines the shape but not the position of the curve.

Theorem 7.3. *Given continuous functions $\sigma : [0,T] \to (0,\infty)$ and $\kappa : [0,T] \to \mathbb{R}$, $p_0 \in \mathbb{E}^2$, and $e_1 \in \dot{\mathbb{E}}^2$, there exists a unique curve $c : [0,T] \to \mathbb{E}^2$ and moving frame*

$$E(t) = \begin{bmatrix} e_1(t) & e_2(t)\end{bmatrix},\ 0 \le t \le T,$$

such that $c(0) = p_0$, $\dot{c} = \sigma e_1$, and E satisfies (7.3).

Corollary 7.4. *Given a continuous function $\kappa : [0,T] \to \mathbb{R}$, if a curve c_1 with moving frame E_1 and another curve c_2 with moving frame E_2 satisfy $|\dot{c}_1| = |\dot{c}_2|$ and* (7.3)*, then there exists a rigid motion $R : \mathbb{E}^2 \to \mathbb{E}^2$ such that $c_2 = R \circ c_1$.*

7.3.5 Frenet-Serret frame in Euclidean 3-space

The following lemma will be useful:

Lemma 7.5. *If $v_1, v_2 : I \to \dot{\mathbb{E}}^m$ are C^1 vector-valued functions satisfying*

$$v_1 \cdot v_2 = 0,$$

then

$$v_1' \cdot v_2 + v_1 \cdot v_2' = 0,$$

which is equivalent to

$$v_2 \cdot v_1' = -v_1 \cdot v_2'. \tag{7.4}$$

Let $c : I \to \mathbb{E}^3$ be a C^2 curve with nonzero velocity $\cdot c$, speed $\sigma = |\dot{c}|$, and

$$e_1 = \frac{\dot{c}}{|\dot{c}|}.$$

Also, assume that $\dot{e}_1$ is nonzero and let e_2 be the unit vector pointing in the direction of $\dot{e}_1$. It follows that there is a nonnegative function κ such that

$$\dot{e}_1 = \sigma\kappa e_2. \tag{7.5}$$

There now exists a C^1 vector-valued function $e_3 : I \to \dot{\mathbb{E}}^3$, unique up to sign, such that, for each $t \in I$,

$$E(t) = \begin{bmatrix} e_1(t) & e_2(t) & e_3(t) \end{bmatrix}$$

is an orthonormal basis of $\dot{\mathbb{E}}^3$. Since $e_3 \cdot \dot{e}_3 = 0$, $\dot{e}_3$ is a linear combination of e_1 and e_2. By Lemma 7.5 and (7.5),

$$e_1 \cdot \dot{e}_3 = -e_3 \cdot \dot{e}_1 = 0.$$

It follows that $\dot{e}_3$ is a scalar multiple of e_2. In other words, there is a function τ such that

$$\dot{e}_3 = -\sigma\tau e_2.$$

From the equations above and Lemma 7.5,

$$e_1 \cdot e_2' = -e_2 \cdot e_1' = -\kappa \text{ and } e_3 \cdot e_2' = -e_2 \cdot e_3' = \tau,$$

and therefore

$$\dot{e}_2 = -\kappa e_1 + \tau e_3.$$

Everything above can be summarized as following: Let $c : I \to \mathbb{E}^3$ be a C^2 curve, where its velocity $v(t)$ and accelerations $a(t)$ are linearly independent for each $t \in I$. Then there exists, for each $t \in I$, an orthonormal frame $E(t)$ such that

$$\begin{aligned} E(t) &= \begin{bmatrix} \dot{e}_1(t) & \dot{e}_2(t) & \dot{e}_3(t) \end{bmatrix} \\ &= \sigma \begin{bmatrix} \kappa e_2(t) & -\kappa e_1(t) + \tau e_3(t) & -\tau e_1(t). \end{bmatrix} \\ &= \sigma \begin{bmatrix} e_1(t) & e_2(t) & e_3(t) \end{bmatrix} \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \end{aligned} \tag{7.6}$$

This frame is called the Frenet-Serret frame.

Theorem 7.6. *Given continuous functions $\sigma : [0, T] \to (0, \infty)$ and $\kappa, \tau : [0, T] \to \mathbb{R}$, $p_0 \in \mathbb{E}^3$, and $e_1 \in \dot{\mathbb{E}}^3$, there exists a unique curve $c : [0, T] \to \mathbb{E}^3$ and moving frame*

$$E(t) = \begin{bmatrix} e_1(t) & e_2(t) & e_3 \end{bmatrix}, \ 0 \le t \le T,$$

such that $c(0) = p_0$, $\dot{c} = \sigma e_1$, and E satisfies (7.6).

Corollary 7.7. *Given continuous functions $\kappa, \tau : [0, T] \to \mathbb{R}$, if a curve c_1 with moving frame E_1 and another curve c_2 with moving frame E_2 in $\mathbb{E}^3$ satisfy $|\dot{c}_1| = |\dot{c}_2|$ and* (7.6)*, then there exists a rigid motion $R : \mathbb{E}^3 \to \mathbb{E}^3$ such that $c_2 = R \circ c_1$.*

Chapter 8

Surfaces in Cartesian 3-space

8.1 Three imperfect definitions of a surface

8.1.1 Graph

Given an open domain $D \subset \mathbb{R}^2$ and a C^1 function $h : D \to \mathbb{R}$, let

$$S = \{z = h(x, y)\} = \{(x, y, h(x, y)) \ : \ (x, y) \in D\}.$$

Examples:

1. The upper half of the unit sphere centered at the origin is

$$S = \{z = \sqrt{1 - x^2 - y^2}, \ x^2 + y^2 \le 1\}.$$

2. The plane that passes through the points $(3, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 1)$ is

$$S = \left\{z = 1 - \frac{x}{3} - \frac{y}{2} \ : \ (x, y) \in \mathbb{R}^2\right\}.$$

3. The graph

$$S = \{z = \sqrt{x^2 + y^2} \ : \ (x, y) \in \mathbb{R}^2\}$$

is an upside cone with its vertex at the origin.

8.1.2 Level set

Given an open set $O \subset \mathbb{R}^3$, a C^1 function $f : O \to \mathbb{R}$, and a scalar constant c, consider the level set

$$S = \{f(x, y, z) = c\} = \{(x, y, z) \in O \ : \ f(x, y, z) = c\}.$$

Examples:

1. The unit sphere centered at the origin is

$$S = \{x^2 + y^2 + z^2 - 1 = 0\}.$$

2. The plane that passes through the points $(3,0,0)$, $(0,2,0)$, and $(0,0,1)$ is the level set

$$S = \left\{\frac{x}{3} + \frac{y}{2} + z = 1\right\}.$$

3. The level set

$$S = \{x^2 + y^2 - z^2 = 0\}$$

   is a double cone with its vertex at the origin.

8.1.3 Parameterized surface

Given a domain $D \subset \mathbb{R}^2$ and a C^1 map $\Phi : D \to \mathbb{R}^3$,

$$S = \Phi(D) = \{(\Phi(u,v)\ :\ (u,v) \in D\} = \{(x(u,v), y(u,v), z(u,v)\ :\ (u,v) \in D\},$$

where

$$\Phi(u,v) = (x(u,v), y(u,v), z(u,v)).$$

Example:

1. The unit sphere centered at the origin can be parameterized by spherical coordinates,

$$S = \{(\sin\phi\cos\theta, \sin\phi, \sin\theta, \cos\phi),\ :\ (\phi,\theta) \in [0,\pi] \times [0,2\pi]\}.$$

2. A cylinder can be parameterized using cylindrical coordinates

$$S = \{(\cos\theta, \sin\theta, z)\ :\ (\theta, z) \in [0,2\pi] \times \mathbb{R}\}.$$

3. A graph of a function h over a domain $D \subset \mathbb{R}^2$ is parameterized by $(x,y) \in D$,

$$S = \{(x, y, h(x,y)\ :\ (x,y) \in D\}.$$

4. A cone can be parameterized using cylindrical coordinates

$$S = \{(r\cos\theta, r\sin\theta, r)\ :\ (r,\theta) \in [0,\infty) \times [0,2\pi]\}.$$

Each of these definitions is flawed. And are they equivalent? Ultimately, we want to define what a surface in affine 3-space is. How can these definitions be used in that setting?

8.2 2-planes in 3-space

The simplest surface in $\mathbb{R}^3$ is a 2-plane. It can be described as follows:

1. **Graph:** Given an affine function

$$h(x,y) = ax + by + c,$$

let

$$S = \{z = h(x,y)\} = \{z = ax + by + c\}.$$

Examples:

$$S = \{z = 0\}$$
$$S = \{z = x + y + 1\}.$$

The only shortcoming is that vertical planes are not graphs. This can be resolved by allowing x or y be the output variable and the other two variables be the input variables.

2. **Level set.** Given an affine function

$$f(x,y,z) = \alpha x + \beta y + \delta z + \gamma,$$

let

$$S = \{f(x,y,z) = 0\} = \{\alpha x + \beta y + \delta z + \gamma = 0\} = \left\{ \begin{bmatrix} \alpha & \beta & \delta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \gamma = 0 \right\}.$$

This, however, is not always a surface, because, if $\alpha = \beta = \delta = 0$, then S is not a 2-plane. An extra assumption is needed, namely $\langle \alpha, \beta, \gamma \rangle \neq 0$. We can write this as follows: First, recall that the differential of f is defined to be

$$df = \partial_x f\, dx + \partial_y f\, dy + \partial_z f\, dz.$$

In order for S to be a plane and therefore a surface, we need to assume that

$$df \neq 0$$

3. **Parameterized surface.** Given an affine map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\Phi(u,v) = (x_0, y_0, z_0) + A\langle u, v \rangle.$$

where $(x_0, y_0, z_0) \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \\ a^3_1 & a^3_2, \end{bmatrix},$$

let

$$\begin{aligned} S &= \{\Phi(u,v) \,:\, \langle u,v\rangle\} \\ &= \{(x_0,y_0,z_0) + A\langle u,v\rangle,\ \langle u,v\rangle \in \mathbb{R}^2\} \\ &= \{(x_0 + a^1_1 u + a^1_2 v, y_0 + a^2_1 u + a^2_2 v, z_0 + a^3_1 u + a^3_2 v)\}. \end{aligned}$$

This, however, does not always work. If the rank of A is not maximal, i.e., 2, then S is either a point or a line. We must therefore assume that the rank of A is maximal. Equivalently, we must assume $\ker A = \{0\}$.

The assumptions needed for the function f and map Φ are nondegeneracy conditions. Notice that they can be expressed in terms of the differentials of f and Φ. In particular, the set $S = \{f = 0\}$ is a surface, if

$$df \neq 0.$$

The set $S = \Phi(\mathbb{R}^2)$ is a surface, if the Jacobian of Φ,

$$\partial\Phi = \begin{bmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \\ \partial_u z & \partial_v z \end{bmatrix}$$

has maximal rank.

8.3 Local descriptions of a surface

To define a surface properly, it suffices to define what a small piece of a surface is. The basic idea is that, given any point p in a surface S, there is a small piece of S containing p that is a local manifold with a good enough parameterization.

8.3.1 Graph

A set $S \subset \mathbb{R}^3$ is a C^k *surface*, if, for any $p \in S$, there exists an open neighborhood $O \subset \mathbb{R}^3$ of p, an open domain $D \subset \mathbb{R}^2$ and a C^k function $h : D \to \mathbb{R}$ such that one of the following holds:

$$\begin{aligned} S \cap O &= \{z = h(x,y) \,:\, (x,y) \in D\} \\ S \cap O &= \{y = h(z,x) \,:\, (x,y) \in D\} \\ S \cap O &= \{x = h(y,z) \,:\, (x,y) \in D\}. \end{aligned}$$

8.3.2 Level set

A set $S \subset \mathbb{R}^3$ is a C^k *surface*, if for each $p_0 = (x_0, y_0, z_0) \in S$, there exists an open $O \subset \mathbb{R}^3$ containing p_0 and a C^k function $f : O \to \mathbb{R}$ such that

$$S \cap O = \{f(x,y,z) = 0\} \text{ and } df(p_0) \neq 0,$$

8.3.3 Parameterized surface

A set $S \subset \mathbb{R}^3$ is a C^k *surface*, if for each $p_0 \in S$, there exists an open domain $D \subset \mathbb{R}^2$, an open set $O \subset \mathbb{R}^3$ containing p_0, and a C^k map $\Phi : D \to \mathbb{R}^3$ such that

$$S \cap O = \{\Phi(u,v) \ : \ (u,v) \in D\}$$

and the Jacobian matrix of Φ at (u_0, v_0), where $\Phi(u_0, v_0) = p_0$, has maximal rank. In other words,

$$\partial\Phi(u_0,v_0) = \begin{bmatrix} \partial_u x(u_0,v_0) & \partial_v x(u_0,v_0) \\ \partial_u y(u_0,v_0) & \partial_v y(u_0,v_0) \\ \partial_u z(u_0,v_0) & \partial_v z(u_0,v_0) \end{bmatrix}$$

has rank 2.

Observe that the map Φ can also be treated as a coordinate map for the local manifold $M = S \cap O$.

8.4 Equivalence of definitions

If S is locally a graph, then, near each $p_0 \in S$,

$$S = \{z = h(x,y)\} = \{f(x,y,z) = 0\} = \{(x,y,z) = \Phi(u,v)\},$$

where

$$\begin{aligned} f(x,y,z) &= z - h(x,y) \\ \Phi(u,v) &= (u,v,h(u,v)). \end{aligned}$$

Therefore, S is locally both a level set and a parameterized surface.

The converse statements require the implicit and inverse function theorems. We state the exactly versions we need here. Proofs can be found in Chapter E of the appendix.

Theorem 8.1. *(Implicit function theorem) Given an open set $O \subset \mathbb{R}^3$, a point $p_0 \in O$, and C^k function $f : O \to \mathbb{R}$ such that*

$$f(p_0) = 0 \text{ and } \partial_z f(p_0) \neq 0,$$

there exists an open neighborhood $N \subset \mathbb{R}^3$ of p_0, an open domain $D \subset \mathbb{R}^2$, and a C^k function $h : D \to \mathbb{R}$ such that, for any $p \in N$,

$$\{f(x,y,z) = 0 \ : \ (x,y,z) \in N\} = \{(x,y,h(x,y)) \ : \ (x,y) \in D\}.$$

Theorem 8.2. *(Inverse function theorem) Given an open domain $D \subset \mathbb{R}^2$, a point $(u_0, v_0) \in D$, and a C^k map $\Phi : D \to \mathbb{R}^3$, let*

$$\begin{aligned} \widehat{\Phi} &: D \to \mathbb{R}^2 \\ (u,v) &\mapsto (x(u,v), y(u,v)). \end{aligned}$$

If the Jacobian matrix $\partial\widehat{\Phi}(u_0,v_0)$ has maximal rank (i.e., is invertible), then there exists an open neighborhood $N \subset D$ of (u_0,v_0) and a C^k map $\Psi : \widehat{\Phi}(N) \to \mathbb{R}^2$, such that, for any $(x,y) \in \widehat{\Phi}(N)$,

$$\Phi(\Psi(x,y)) = (x,y).$$

8.5 Tangent space at a point on a surface

Given $p \in S$, let $O \subset \mathbb{R}^3$ be an open neighborhood of p such that $M = S \cap O$ is a local $C^i k$ manifold. The tangent space at p is defined to be T_pM and denoted by T_pS. Recall that it consists of all possible velocities at p of curves that lie in the surface and pass through p. Therefore, given $v \in T_pS$, there exists a C^1 curve $c : (-\delta, \delta) \to S$ such that $c(0) = p$ and $\dot{c}(0) = v$. On other hand, since $S \subset \mathbb{R}^3$, c is also a parameterized curve in $\mathbb{R}^3$ and therefore $\dot{c}(0) \in \mathbb{R}^3$. To see how the two different definitions of $\dot{c}$ are related, we use a coordinate map.

In particular, suppose D is an open neighborhood of $(0,0) \in \mathbb{R}^2$ and $\Phi : D \to M = S \cap O$ is a coordinate map, where $\Phi(0) = p \in S$. Given any $v \in T_pS$, there is a C^1 curve $c : (-\delta, \delta) \to S$ such that $c(0) = p$ and $\dot{c} = v$. This is equivalent to saying that the curve $\hat{c} = \Phi\,(-\delta, \delta) \to D$, where $\hat{c}(0) = (0,0)$, $c = \Phi \circ \hat{c} : (-T, T) \to S$ is a C^1 curve in S. Moreover, since

$$c(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))),$$

The velocity of c at p_0 is

$$\begin{aligned}
\dot{c}(0) &= \left\langle \frac{\partial x}{\partial u}\dot{u} + \frac{\partial x}{\partial v}\dot{v}, \frac{\partial y}{\partial u}\dot{u} + \frac{\partial y}{\partial v}\dot{v}, \frac{\partial z}{\partial u}\dot{u} + \frac{\partial z}{\partial v}\dot{v} \right\rangle \\
&= \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} \\
&= \partial\Phi(p_0)\langle \dot{u}, \dot{v} \rangle
\end{aligned}$$

where $\langle \dot{u}, \dot{v} \rangle = \hat{c}'(0)$. Therefore, the Jacobian of Φ defines at each $p_0 \in S$, a linear map

$$\begin{aligned}
\partial\Phi(p_0) : \mathbb{R}^2 &\to T_{p_0}S \subset \mathbb{R}^3 \\
\langle \dot{u}, \dot{v} \rangle &\mapsto \partial\Phi(p_0)\langle \dot{u}, \dot{v} \rangle
\end{aligned} \tag{8.1}$$

Since $\partial\Phi(p_0)$ has maximal rank, this map is injective, and therefore $T_{p_0}S$ is a 2-dimensional linear subspace of $\mathbb{R}^3$.

Conversely, since Φ is an injective map, given any C^1 curve $c : (-T, T) \to S$, there is a corresponding curve $\hat{c} = \Phi^{-1} \circ c : (-T, T) \to D$. One can use the inverse function theorem to prove that $\hat{c}$ is C^1. The argument above shows that $\dot{c}(0) \in T_pS$ lies in the image of the linear map $\partial\Phi(p_0)$. This implies that the map (8.1) is a linear isomorphism.

The tangent space T_p can also described using the definition of the surface S as a level set. Suppose

$$S = \{p \in O \,:\, f(p) = 0\},$$

where O is an open neighborhood of p_0 in $\mathbb{R}^3$ and f is a C^1 function on O such that $df \neq 0$. Given any C^1 curve $c : (-T, T) \to S \cap O$ such that $c(0) = p_0$ and $\dot{c}(0) = \langle \dot{x}, \dot{y}, \dot{z} \rangle$,

$$0 = \frac{d}{dt} f(c(t)) = \frac{d}{dt}(f(x(t), y(t), z(t))) = \dot{x}\partial_x f + \dot{y}\partial_y f + \dot{z}\partial_z f = \nabla f \cdot \dot{c}. \tag{8.2}$$

It follows that

$$T_pS \subset \{v \in \mathbb{R}^3 \ : \ v \cdot \nabla f(p)\rangle = 0\}.$$

However, since both sides are 2-dimensional linear subpsaces, they must be the same. Therefore, they are in fact equal.

8.6 Alternative notation

Using the letters u, v, w, x, y, z can sometimes be confusing. Also, this cannot be used easily in higher dimensions. We will somtimes use integer-valued subscripts and superscripts, as explained here.

8.6.1 Coordinates and indices

We will denote the coordinates of a general point in $\mathbb{R}^3$ by

$$y = (y^1, y^2, y^3)$$

and the corresponding indices by $1 \le a, b, c \le 3$.

Given a surface $S \subset \mathbb{R}^3$ and coordinate map $\Phi : D \to S$, where $D \subset \mathbb{R}^2$, we will denote a typical point in D by

$$x = (x^1, x^2)$$

and the corresponding indices by $1 \le i, jk \le 2$. The map Φ will also be denoted by

$$y(x) = (y^1(x^1, x^2), y^2(x^1, x^2), y^3(x^1, x^2)).$$

In particular, the letter x indicates a point in the domain of the parameterization and y a point in the range.

8.6.2 Partial derivatives

Given a function $f : O \to \mathbb{R}$, where O is a open subset of $\mathbb{R}^3$, its partial derivatives will be denoted

$$\partial_a f = \frac{\partial f}{\partial y^a},\ a = 1, 2, 3.$$

The list of all possible first derivatives will be viewed as a covector and therefore written as a row matrix of functions,

$$\partial f = \begin{bmatrix} \partial_1 f & \partial_2 f & \partial_3 f \end{bmatrix}.$$

Second order partial derivatives will be denoted

$$\partial^2_{ab} f = \partial^2_{ba} f = \frac{\partial^2 f}{\partial y^a \partial y^b},$$

Given a function $f : D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^2$, its partial derivatives will be denoted

$$\partial_i f = \frac{\partial f}{\partial x^i}.$$

Second order partial derivatives will be denoted

$$\partial^2_{ij} f = \partial^2_{ji} f = \frac{\partial^2 f}{\partial y^i \partial y^j}.$$

The list of all second order partial derivatives will be written as a matrix, called the Hessian of f and denoted by

$$\partial^2 f = \begin{bmatrix} \partial^2_{11} f & \partial^2_{12} f \\ \partial^2_{21} f & \partial^2_{22} f. \end{bmatrix}.$$

The first index indicates the row and second index the column of each entry in the matrix. Since partials commute, the matrix is symmetric.

Given an $\mathbb{R}^m$-valued function of $x \in D$,

$$v(x) = (v^1(x^1, x^2), \ldots, v^m(x^1, x^2)),$$

the partial derivatives form a matrix, which will be denoted

$$\frac{\partial v}{\partial x} = \begin{bmatrix} \partial_1 v^1 & \partial_2 v^1 \\ \vdots & \vdots \\ \partial_1 v^m & \partial_2 v^m \end{bmatrix} = \begin{bmatrix} \frac{\partial v^1}{\partial x^1} & \frac{\partial v^1}{\partial x^2} \\ \vdots & \vdots \\ \frac{\partial v^m}{\partial x^1} & \frac{\partial v^m}{\partial x^2}. \end{bmatrix}$$

Here, the superscript indicates the row of the matrix and the subscript indicates the column. Similar notation is used for an $\mathbb{R}^m$-valued function of $y \in O$. This matrix is sometimes called the Jacobian of v.

8.7 Surface with boundary

Denote the upper half plane by

$$\mathbb{H}^2 = \{(x^1, x^2) \; : \; x^2 \geq 0\}.$$

A set $S \subset \mathbb{R}^3$ is a C^k *surface with boundary*, if for any $p \in S$, there exists an open neighborhood $O \subset \mathbb{R}^3$ of p, an open neighborhood $D \subset \mathbb{R}^2$ of a point $x_0 \in \mathbb{H}^2$, and a bijective C^k map $\Phi : D \cap \mathbb{H}^2 \to S \cap O$ such $\Phi(x_0) = p$ and $\partial\Phi(x)$ has maximal rank for all $x \in D \cap \mathbb{H}^2$.

Chapter 9

Surface in Affine 3-space

A set $S \subset \mathbb{A}^3$ is a surface if, given an affine basis P of $\mathbb{A}^3$, the set $I_P^{-1}(S) \subset \mathbb{R}^3$ is a surface as defined in the previous chapter.

A more abstract equivalent definition is the following. Note first that $\mathbb{A}^3$ is a local C^∞ manifold.

Definition 9.1. A set $S \subset \mathbb{A}^3$ is a C^k *surface* if, for any $p \in S$, there is an open set $O \subset \mathbb{A}^3$ containing p such that $M = S \cap O$ is a local manifold and the map $f : M \to \mathbb{A}^3$ is a C^k map of local manifolds.

There is a natural linear injection

$$\begin{aligned} T_pS &\to \mathbb{V}^3 \\ v &\mapsto c'(0), \end{aligned}$$

where $c : I \to S \subset \mathbb{A}^3$ is a C^1 curve such that $c(0) = p$ and $c'(0) = v$. In particular T_pS is a linear subspace of $\mathbb{V}^3$.

Chapter 10

Surfaces in Euclidean 3-space

In the previous chapter, we never used any geometric concepts such as distance, length, or angle. In this chapter, we use these geometric quantities to study the geometry of a surface in Euclidean 3-space $\mathbb{E}^2$.

10.1 First fundamental form

The first fundamental form is simply the dot product restricted to vectors in the subspace $T_pS \subset \dot{\mathbb{E}}^3$. We want, however, to study using a coordinate map $\Phi : D \to S \subset \mathbb{E}^3$. In particular, we want to "pull back" the dot product on $\mathbb{R}^3$ to the vector space T_pS.

First, recall that the dot product of any two vectors $\langle \dot{x}_1, \dot{y}_1, \dot{z}_1 \rangle$ and $\langle \dot{x}_2, \dot{y}_2, \dot{z}_2 \rangle$ is

$$\langle \dot{x}_1, \dot{y}_1, \dot{z}_1 \rangle \cdot \langle \dot{x}_2, \dot{y}_2, \dot{z}_2 \rangle = \dot{x}_1 \dot{x}_2 + \dot{y}_1 \dot{y}_2 + \dot{z}_1 \dot{z}_2.$$

Therefore, given any two $\hat{c}_1, \hat{c}_2 : (-T, T) \to D$ such that $\hat{c}_1(0) = \hat{c}_2(0) = p_0 \in S$, the dot product of the velocities $\dot{c}_1 = (\Phi \circ \hat{c}_1)'(0)$ and $\dot{c}_2 = (\Phi \circ \hat{c}_2)'(0)$ at p_0 is given by

$$\begin{aligned}
\dot{c}_1(0) \cdot \dot{c}_2(0) &= (\Phi \circ \hat{c}_1)'(0) \cdot (\Phi \circ \hat{c}_2)'(0) \\
&= (\partial\Phi \hat{c}_1'(0)) \cdot (\partial\Phi \hat{c}_2'(0)) \\
&= \left(\frac{\partial x}{\partial u}\dot{u}_1 + \frac{\partial x}{\partial v}\dot{v}_1\right)\left(\frac{\partial x}{\partial u}\dot{u}_2 + \frac{\partial x}{\partial v}\dot{v}_2\right) + \left(\frac{\partial y}{\partial u}\dot{u}_1 + \frac{\partial y}{\partial v}\dot{v}_1\right)\left(\frac{\partial y}{\partial u}\dot{u}_2 + \frac{\partial y}{\partial v}\dot{v}_2\right) \\
&\quad + \left(\frac{\partial z}{\partial u}\dot{u}_1 + \frac{\partial z}{\partial v}\dot{v}_1)(\frac{\partial z}{\partial u}\dot{u}_2 + \frac{\partial z}{\partial v}\dot{v}_2\right) \\
&= \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial u}\right)\dot{u}_1\dot{u}_2 \\
&\quad + \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial z}{\partial v}\frac{\partial z}{\partial u}\right)(\dot{u}_1\dot{v}_2 + \dot{u}_2\dot{v}_1) \\
&\quad + \left(\frac{\partial x}{\partial v}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial v}\frac{\partial z}{\partial v}\right)(\dot{v})^2 \\
&= \begin{bmatrix} \dot{u}_1 & \dot{v}_1 \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{bmatrix} \dot{u}_2 \\ \dot{v}_2 \end{bmatrix} \\
&= \langle \dot{u}_1, \dot{v}_1 \rangle \cdot (\partial\Phi)^T (\partial\Phi) \langle \dot{u}_2, \dot{v}_2 \rangle,
\end{aligned}$$

where $(\partial\Phi)^T$ denotes the transpose of the matrix $\partial\Phi$.

Therefore, if, for each $p \in S$, we define the symmetric 2-by-2 matrix

$$G(p) = (\partial\Phi(p))^T (\partial\Phi(p)),$$

and, given any tangent vectors $\langle \dot{u}_1, \dot{v}_1 \rangle, \langle \dot{u}_2, \dot{v}_2 \rangle \in T_pS$, define

$$g(p)(\langle \dot{u}_1, \dot{v}_1 \rangle, \langle \dot{u}_2, \dot{v}_2 \rangle) = \langle \dot{u}_1, \dot{v}_1 \rangle \cdot G \langle \dot{u}_2, \dot{v}_2 \rangle,$$

then,

$$(\partial\Phi(p)\langle \dot{u}_1, \dot{v}_1 \rangle) \cdot ((\partial\Phi(p))\langle \dot{u}_2, \dot{v}_2 \rangle) = g(p)(\langle \dot{u}_1, \dot{v}_1 \rangle, \langle \dot{u}_2, \dot{v}_2 \rangle).$$

10.2 Change of notation

The calculation above is a mess, using several different variables. We introduce new notation that is easier to write and has the advantage that it makes generalization to higher dimensions easy.

From now on, a point in $\widetilde{\mathbb{R}}^2$ will be denoted $x = (x^1, x^2)$, and a vector in $\widehat{\mathbb{R}}^2$ by $\dot{x} = \langle \dot{x}^1, \dot{x}^2 \rangle$. A point in $\widetilde{\mathbb{R}}^3$ will be denoted $y = (y^1, y^2, y^3)$, and a vector in $\widehat{\mathbb{R}}^3$ by $\langle \dot{y}^1, \dot{y}^2, \dot{y}^3 \rangle$. A coordinate chart will be denoted $y(x) = (y^1(x^1, x^2), y^2(x^1, x^2), y^3(x^1, x^2))$.

If we fix x^2, then the map $t \mapsto y(t, x^2)$ defines a curve in S. This defines a family of curves on the coordinate chart. There is a second family of curves defined by fixing x^1 and using x^2 as the parameter of a curve. The velocities of these curves are the partial derivatives of y,

$$\frac{\partial y}{\partial x^1} \text{ and } \frac{\partial y}{\partial x^2},$$

which are the columns of the Jacobian

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} \\ \frac{\partial y^3}{\partial x^1} & \frac{\partial y^3}{\partial x^2} \end{bmatrix}.$$

That the Jacobian has rank 2 implies that

$$\left(\frac{\partial y}{\partial x^1}, \frac{\partial y}{\partial x^2}\right)$$

is, for each $p \in S$, a basis of T_pS. Morevoer, the dot product of

$$\begin{aligned} v_1 &= v_1^1 \frac{\partial y}{\partial x^1} + v_1^2 \frac{\partial y}{\partial x^2} \\ v_2 &= v_2^1 \frac{\partial y}{\partial x^1} + v_2^2 \frac{\partial y}{\partial x^2}, \end{aligned}$$

is given by

$$\begin{aligned} v_1 \cdot v_2 &= \left(v_1^1 \frac{\partial y}{\partial x^1} + v_1^2 \frac{\partial y}{\partial x^2}\right) \cdot \left(v_2^1 \frac{\partial y}{\partial x^1} + v_2^2 \frac{\partial y}{\partial x^2}\right) \\ &= g_{ij}(p) v_1^i v_2 v^j, \end{aligned}$$

where, for each $1 \le i, j \le 2$,

$$g_{ij} = \frac{\partial y}{\partial x^i} \cdot \frac{\partial y}{\partial x^j}. \tag{10.1}$$

It follows that the length of any tangent vector or curve at $p \in S$ and the angle of any pair of vectors in T_pS can be determined using the coordinates (x^1, x^2) and the symmetric positive definite matrix $G = [g_{ij}(p)]$.

More generally, an inner product on T_pS, for each $p \in S$, is called a *Riemannian metric*. Given a Riemannian metric g, $p \in S$, and a basis $e_1, \ldots, e_m \in T_pS$, there is a positive definite matrix $G(p) = [g_{ij}(p)]$, where

$$g_{ij}(p) = e_i \cdot e_j.$$

Such a Riemannian metric need not be the same as the one given by (10.1). Moreover, given a Riemannian metric g, there can be more than one embedding that satisfies (10.1). For these reasons, we call a Riemannian metric an intrinsic geometric structure on S.

10.3 Gauss map

If a surface $S \subset \widetilde{\mathbb{R}}$ is defined as the level set of a C^2 function f, then the Gauss map is defined to be

$$\begin{aligned} \eta : S &\to \widehat{\mathbb{R}}^3 \\ p &\mapsto \frac{\nabla f(p)}{|\nabla f(p)|}, \end{aligned}$$

For each $p \in S$, the vector $\eta(p)$ is a unit vector orthogonal to T_pS. It is unique up to sign.

10.4 Geometry of a curve on a surface

Recall that the curvature of a curve in $\mathbb{E}^2$ and the curvature and torsion of a unit speed curve in $\mathbb{E}^3$ uniquely determine the shape of the curve. Moreover, the definition of these functions does not depend on how the curve is described (i.e., the parameterization of the curve or the coordinates in Euclidean space). The analogous geometric invariants for a surface in $\mathbb{E}^3$ are called the *first and second fundamental forms*. The first fundamental form plays a role similar to the unit speed parameterization of a curve. The second fundamental form is analogous to the curvature and torsion of a curve. We will see later that the first and second fundamental forms uniquely determine the shape of the surface.

Before describing the second fundamental form of a surface S, we note that the *first fundamental form* at each point $p \in S$ is simply the dot product on $\dot{\mathbb{E}}^3$ but restricted to vectors tangent to S at p. Its

Given a C^2 surface $S \subset \widetilde{\mathbb{R}}^3$, a point $p \in S$, and a unit tangent vector $u \in T_pS$, consider a C^2 unit speed curve $c : I \to S$ such that $c(0) = p$ and $\dot{c}(0) = u$. We can define an orthonormal frame (e_1, e_2, e_3) along c, where

$$\begin{aligned} e_1(t) &= \dot{c}(t) \\ e_3(t) &= \eta(c(t)). \end{aligned}$$

This is *not* the Frenet-Serret frame, and therefore we have to recompute the derivatives of e_1, e_2, e_3 with respect to t.

First, since e_1 is unit, it follows by (7.1) that $\dot{e}_1$ is orthogonal to e_1 and, therefore, there are functions κ_2 and κ_3 such that

$$\ddot{c}(t) = \dot{e}_1(t) = \kappa_2(t)e_2(t) - \kappa_3(t)e_3(t).$$

Since e_2 is tangent to S and e_3 is normal to S, we call κ_2 the tangential curvature and κ_3 the normal curvature to the curve c, relative to the surface S. The tangential curvature measures how quickly e_1 is twisting in the tangent plane, and the normal curvature measures how quickly e_1 is twisting out of the tangent plane.

If S is the level set of a C^2 function f, then, differentiating (8.2), we get

$$\begin{aligned}
0 &= \frac{d^2}{dt^2} f(x(t), y(t), z(t)) \\
&= \frac{d}{dt}(\langle \partial_x f, \partial_y f, \partial_z f\rangle \cdot \langle \dot{x}, \dot{y}, \dot{z}\rangle \\
&= \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix} \begin{bmatrix} \partial^2_{xx} f & \partial^2_{xy} f & \partial^2_{xz} f \\ \partial^2_{yx} f & \partial^2_{yy} f & \partial^2_{yz} f \\ \partial^2_{zx} f & \partial^2_{zy} f & \partial^2_{zz} f \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + \langle \partial_x f, \partial_y f, \partial_z f\rangle \cdot \langle \ddot{x}, \ddot{y}, \ddot{z}\rangle \qquad (10.2) \\
&= \dot{c} \cdot (\partial^2 f)\dot{c} + \nabla f \cdot \ddot{c} \\
&= \cdot c \cdot (\partial^2 f)\dot{c} - |\nabla f| e_3 \cdot (\kappa_2 e_2 + \kappa_3 e_3) \\
&= \cdot c \cdot (\partial^2 f)\dot{c} - |\nabla f| e_3
\end{aligned}$$

where

$$\partial^2 f = \begin{bmatrix} \partial^2_{xx} f & \partial^2_{xy} f & \partial^2_{xz} f \\ \partial^2_{yx} f & \partial^2_{yy} f & \partial^2_{yz} f \\ \partial^2_{zx} f & \partial^2_{zy} f & \partial^2_{zz} f \end{bmatrix}$$

is the Hessian of f. Therefore, setting $t = 0$, we get

$$\kappa_3(0) = -u \cdot H(p)u,$$

where

$$H = \frac{\partial^2 f}{|\nabla f|}$$

is a 2-by-2 symmetric matrix. This shows that the normal curvature at each point of the curve depends only on the direction of the curve at that point. It does not depend on the shape of the curve at all.

10.5 Second fundamental form

We can also show that, for any $v \in T_p S$, $v \cdot Hv$ depends only on the surface S and not on the function f as follows: If S is also the level set of another C^2 function g, then, by Lemma **??**, there exists a C^2 function ϕ such that $g = \phi f$. Flipping the sign of g, if necessary, we can assume that, along S, ∇g points in the same direction as ∇f. That implies $\phi > 0$. Therefore, along S,

$$\begin{aligned}
\frac{\partial^2 g}{|\nabla g|} &= \frac{\partial^2 g}{|\nabla g|} \\
&= \frac{\partial^2(\phi f)}{|\nabla(\phi f)|} \\
&= \frac{\phi \partial^2 f + \nabla\phi \otimes \nabla f + \nabla f \otimes \nabla\phi + f\partial^2\phi}{|\phi\nabla f + f\nabla\phi|} \\
&= \frac{\phi \partial^2 f + \nabla\phi \otimes \nabla f + \nabla f \otimes \nabla\phi}{\phi|\nabla f|},
\end{aligned}$$

where we use the following notation: Given functions f and g,

$$\nabla f \otimes \nabla g = \begin{bmatrix} \partial_x f \partial_x g & \partial_x f \partial_y g & \partial_x f \partial_z g \\ \partial_y f \partial_x g & \partial_y f \partial_y g & \partial_y f \partial_z g \\ \partial_z f \partial_x g & \partial_z f \partial_y g & \partial_z f \partial_z g \end{bmatrix}.$$

Therefore, if $p \in S$ and $v \in T_pS$, then $f(p) = 0$ and $v \cdot \nabla f = 0$. It follows that

$$\begin{aligned} \frac{v \cdot (\partial^2 g) v}{|\nabla g|} &= \frac{v \cdot (\phi \partial^2 f + \nabla \phi \otimes \nabla f + \nabla f \otimes \nabla \phi) v}{\phi |\nabla f|} \\ &= \frac{v \cdot \partial^2 f v + 2(v \cdot \nabla \phi)(v \cdot \nabla f)}{|\nabla f|} \end{aligned}$$

We can therefore define, for each p, a bilinear function

$$\begin{aligned} H(p) : T_pS \times T_pS &\to \mathbb{R} \\ (v, w) &\mapsto \frac{v \cdot (\partial^2 f) w}{|\nabla f|}. \end{aligned}$$

This is called the *second fundamental form* of the surface S at $p \in S$. It is a geometric invariant of S that measures how curved the surface is. In particular, if c is a unit speed curve on S, then its normal curvature at a point $c(t)$ is

$$\kappa_3 = -v \cdot H v,$$

where $v = \dot{c}(t)$. This implies that how quickly the direction of c twists out of the tangent plane of S depends only on the geometry of S and the direction of c.

Given any orthonormal basis of T_pS, $H(p)$ can be written as a symmetric 2-by-2 matrix, which always has 2 real eigenvalues. These are called the principal curvatures. They represent the maximum and minimum possible values of the normal curvature at a point.

Example. The sphere of radius r centered at the origin is the level set $f = r^2$ for the function

$$f(x, y, z) = x^2 + y^2 + z^2.$$

The gradient of f is

$$\nabla f = 2\langle x, y, z\rangle,$$

and therefore

$$\begin{aligned} |\nabla f| &= 2\sqrt{x^2 + y^2 + z^2} \\ &= 2r \\ \frac{\partial^2 f}{|\nabla f|} &= \frac{1}{2r} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \frac{1}{r} I, \end{aligned}$$

where I is the 3-by-3 identity matrix. Since

$$v \cdot \left(\frac{1}{r} I\right) v = \frac{|v|^2}{r} = \frac{1}{r},$$

for any unit vector v, it follows that the principal curvatures are both r^{-1}.

Example. Consider the 1–sheeted hyperboloid

$$S = \{(x, y, z) \in \widetilde{\mathbb{R}}^3 \;:\; x^2 + y^2 - z^2 = a^2\}.$$

This can be written in cylindrical coordinates as

$$S = \{(r\cos\theta,\; r\sin\theta, z) \;:\; r^2 - z^2 = a^2\},$$

where $r^2 = x^2 + y^2$. It is the level set $f = a^2$ for the function $f(x, y, z) = x^2 + y^2 - z^2$. Therefore,

$$\begin{aligned}
\nabla f &= 2\langle x, y, -z\rangle = 2\langle r\cos\theta, r\sin\theta, -z\rangle \\
|\nabla f| &= 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{r^2 + z^2} = 2\sqrt{a^2 + 2z^2} \\
\frac{\partial^2 f}{|\nabla f|} &= \frac{1}{\sqrt{a^2 + 2z^2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\end{aligned}$$

One orthonormal basis of T_pS is

$$e_1 = \langle -\sin\theta, \cos\theta, 0\rangle,\; e_2 = \frac{\langle z\cos\theta, z\sin\theta, r\rangle}{\sqrt{r^2 + z^2}} = \frac{\langle z\cos\theta, z\sin\theta, r\rangle}{\sqrt{a^2 + 2z^2}}$$

Therefore, the second fundamental form of S is given by

$$\begin{aligned}
H(e_1, e_1) &= \frac{1}{\sqrt{a^2 + 2z^2}} \begin{bmatrix} -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{a^2 + 2z^2}} w \\
H(e_1, e_2) = H(e_2, e_1) &= \frac{1}{a^2 + 2z^2} \begin{bmatrix} z\cos\theta & z\sin\theta & r \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \\
&= 0 \\
H(e_2, e_2) &= \frac{1}{(a^2 + z^2)^{3/2}} \begin{bmatrix} z\cos\theta & z\sin\theta & r \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z\cos\theta \\ z\sin\theta \\ r \end{bmatrix} \\
&= \frac{z^2 - r^2}{(a^2 + 2z^2)^{3/2}} \\
&= \frac{-a^2}{(a^2 + 2z^2)^{3/2}}.
\end{aligned}$$

It follows that the principal curvatures are

$$\frac{1}{\sqrt{a^2+2z^2}} \text{ and } \frac{-a^2}{(a^2+2z^2)^{3/2}}.$$

Example. More generally, consider a surface of revolution

$$S = \{(r(z)\cos\theta, r(z)\sin\theta, z)\},$$

where $r : \mathbb{R} \to (0,\infty)$ is the given radial profile function. This is the level set $f = 0$ of the function

$$f(x,y,z) = \frac{1}{2}(x^2 + y^2 - (r(z))^2).$$

Therefore,

$$\begin{aligned}
\partial f &= \langle x, y, -r r_z \rangle \\
|\partial f| &= \sqrt{x^2 + y^2 + (r r_z)^2} \\
&= r\sqrt{1 + r_z^2} \\
\partial^2 f &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r r_{zz} - (r_z)^2 \end{bmatrix} \\
\frac{\partial^2 f}{|\partial f|} &= \frac{1}{r\sqrt{1+r_z^2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r r_{zz} - (r_z)^2 \end{bmatrix} \\
&= \frac{1}{r\sqrt{1+r_z^2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r r_{zz} - (r_z)^2 \end{bmatrix}.
\end{aligned}$$

An orthonormal basis of S is given by

$$\begin{aligned}
e_1 &= \frac{\langle -y, x, 0 \rangle}{\sqrt{x^2+y^2}} \\
&= \frac{\langle -y, x, 0 \rangle}{r} \\
e_2 &= \frac{\langle r_z x, r_z y, r \rangle}{r\sqrt{1+r_z^2}}.
\end{aligned}$$

Therefore, the second fundamental form of S is given by

$$
\begin{aligned}
H(e_1,e_1) &= \frac{1}{r^3\sqrt{1+r_z^2}} \begin{bmatrix} -y & x & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r r_{zz} - (r_z)^2 \end{bmatrix} \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} \\
&= \frac{1}{r\sqrt{1+r_z^2}} \\
H(e_1,e_2) = H(e_2,e_1) &= \frac{1}{r^3(1+r_z^2)} \begin{bmatrix} r_z x & r_z y & r \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r r_{zz} - (r_z)^2 \end{bmatrix} \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} \\
&= 0 \\
H(e_2,e_2) &= \frac{1}{r^3(1+r_z^2)^{3/2}} \begin{bmatrix} r_z x & r_z y & r \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r r_{zz} - (r_z)^2 \end{bmatrix} \begin{bmatrix} r_z x \\ r_z y \\ r \end{bmatrix} \\
&= \frac{(r r_z)^2 - r^2(r r_{zz} + (r_z)^2)}{r^3(1+r_z^2)^{3/2}} \\
&= \frac{-r_{zz}}{(1+r_z^2)^{3/2}}.
\end{aligned}
$$

Chapter 11

Differential forms

11.1 Vector fields and differential 1-forms

Given an open subset $O \subset \mathbb{A}^m$, where $\mathbb{A}^m$ is the affine space of points, a *vector field* is a C^1 map $F : O \to \mathbb{V}^m$, where $\mathbb{V}^m$ is the vector space associated with $\mathbb{A}^m$. If $\mathbb{A}^m = \mathbb{R}^m$, then a vector field $F : O \subset \mathbb{R}^m$ is of the form

$$F(x) = (F^1(x^1, \ldots, x^m), \ldots, F^m(x^1, \ldots, x^m)), \ x \in O.$$

A *differential* 1*-form* is a C^0 map $\theta : O \to (\mathbb{V}^m)^*$. Therefore, for each $x \in O$, $\theta(x)$ is a linear function on $\mathbb{V}^m$.

The basic example of a differential form is the differential of a C^1 function $f : O \to \mathbb{R}$. Recall that the differential of f is a map $df : O \to (\mathbb{V}^m)^*$, where

$$df(x)v = \left.\frac{d}{dt}\right|_{t=0} f(x + tv)$$

is the directional derivative of f along a curve through x with velocity v. If $\mathbb{A}^m = \mathbb{V}^m = \mathbb{R}^m$, then $v = v^i e_i$ and, by the chain rule, we get the familiar formula for the directional derivative,

$$df(x)v = v^i \frac{\partial f}{\partial x^i}(x).$$

11.2 Coordinate differential 1-forms

Each coordinate on $\mathbb{R}^m$ defines a function

$$\begin{aligned} x^i : \mathbb{R}^m &\to \mathbb{R} \\ (p^1, \ldots, p^m) &\mapsto p^i \end{aligned}$$

and corresponding differential 1-forms dx^i, which satisfy

$$dx^i(x)v = v^i.$$

In other words, $dx^1, \ldots, dx^m$ are the dual basis to the standard basis $e_1, \ldots, e_m$.

11.3 Differential k-forms

More generally, a *differential k-form* is a C^0 map $\Theta : O \to \bigwedge^k(\mathbb{V}^m)^*$. Therefore, for each $x \in O$, $\Theta(x)$ is an alternating multilinear function of k vectors in $\mathbb{V}^m$.

The basic example of a differential k-form is the wedge product of k differential 1-forms,

$$\Theta = \theta^1 \wedge \cdots \wedge \theta^k.$$

A special case of this is when the 1-forms are differentials of functions,

$$df^1 \wedge \cdots \wedge df^k,$$

If $\mathbb{A}^m = \mathbb{V}^m = \mathbb{R}^m$, then specific examples are, given any integers $1 \le i_1, \ldots, i_k \le m$,

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Morever, C^0 differential k-form on $O \subset \mathbb{R}^m$ can be written in the form

$$\Theta = \frac{1}{k!} \sum_{1 \le i_1, \cdots, i_k \le m} a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots dx^{i_k},$$

where each $a_{i_1 \cdots i_k}$ is a C^0 function on O and, for each $\sigma \in S_k$,

$$a_{i_{\sigma(1)} \cdots i_{\sigma(k)}} = (-1)^\sigma a_{i_1 \cdots i_k}.$$

11.4 Pullback of a function

The simplest example is how a function on the range of a map can be pulled back to a function on the domain of the map using composition. In other words, given a map $\Phi : \mathbb{A}^m \to \mathbb{B}^n$ and a function $\mathbb{B}^n \to \mathbb{R}$, the pullback of f by Φ is simply the function $\Phi^* f = f \circ \Phi$.

Specifically, let $\mathbb{A}^m$ and $\mathbb{B}^n$ be affine spaces of points, $O \subset \mathbb{A}^m$ and $O' \subset \mathbb{B}^n$ be open sets, and $\Phi : O \to O'$ a C^1 map. Then the pullback of a function $f : O' \to \mathbb{R}$ by Φ is the function

$$\begin{aligned} \Phi^* f : O &\to \mathbb{R} \\ x &\mapsto f(\Phi(x)). \end{aligned}$$

11.5 Pullback of a differential 1-form

The pullback of a differential form is essentially the derivative of the composition of a function with a map. Computing it therefore uses the chain rule. It is called pulling back because a differential form on the range of the map is pulled back to a differential form on the domain of the map.

Recall that the differential of a C^1 map $\Phi : O \to O'$ defines for each $x \in O$ a linear map

$$d\Phi(x) : \mathbb{V}^m \to \mathbb{W}^n,$$

where $\mathbb{V}^m$ and $\mathbb{W}^n$ are the vector spaces associated with $\mathbb{A}^m$ and $\mathbb{B}^n$.

If θ is a C^0 differential 1-form on O', then, for each $y \in O'$, it is a linear function

$$\theta(y) : \mathbb{W}^n \to \mathbb{R}.$$

We can therefore define the pullback of θ by composing the linear function $\theta(\Phi(x)) : \mathbb{W}^n \to \mathbb{R}$ with the linear map $d\Phi(x) : \mathbb{V}^m \to \mathbb{W}^n$. Specifically, the pullback of θ by the map Φ is the 1-form $\Phi^*\theta$ on O such that, for any $x \in O$ and $v \in \mathbb{V}^m$,

$$(\Phi^*\theta)(x)\,v = \theta(\Phi(x))\,d\Phi(x)\,v.$$

If $\mathbb{A}^m = \mathbb{V}^m = \mathbb{R}^m$ and $\mathbb{B}^n = \mathbb{W}^n = \mathbb{R}^n$,

$$\Phi(x) = (y^1(x^1,\dots,x^m),\dots,y^n(x^1,\dots,x^m)),$$

then

$$\Phi^*\,dy^\alpha = \frac{\partial y^\alpha}{\partial x^i}\,dx^i.$$

and, if

$$\theta(y) = a_\alpha(y)\,dy^\alpha,$$

then

$$(\Phi^*\theta)(x) = a_\alpha(y(x))\frac{\partial y^\alpha}{\partial x^i}dx^i.$$

11.6 Pullback of a differential k-form

More generally, the pullback of a C^0 differential k-form on $O' \subset \mathbb{B}^n$ by a C^1 map $\Phi : O \to O'$ is defined by

$$(\Phi^*\Theta)(x)(v_1,\dots,v_k) = \Theta(\Phi(x))(d\Phi(x)v_1,\dots,d\Phi(x)v_k).$$

If $\mathbb{A}^m = \mathbb{V}^m = \mathbb{R}^m$ and $\mathbb{B}^n = \mathbb{W}^n = \mathbb{R}^n$,

$$\Phi(x) = (y^1(x^1,\dots,x^m),\dots,y^n(x^1,\dots,x^m)),$$

and

$$\Theta(y) = \sum_{1\le\alpha_1,\dots,\alpha_k\le n} a_{\alpha_1\cdots\alpha_k}(y)\,dy^{\alpha_1}\wedge\cdots dy^{\alpha_k},$$

then

$$(\Phi^*\Theta)(x) = \sum_{1\le\alpha_1,\dots,\alpha_k\le n} a_{\alpha_1\cdots\alpha_k}(y(x))\left(\frac{\partial y^{\alpha_1}}{\partial x^j}dx^j\right)\wedge\cdots\wedge\left(\frac{\partial y^{\alpha_k}}{\partial x^j}dx^j\right).$$

In particular, if $n = m$ and Θ is an m-form, then

$$(\Phi^*\Theta)(x) = \left(\det\frac{\partial y}{\partial x}\right)dx^1\wedge\cdots\wedge dx^m.$$

Chapter 12

Integration

12.1 Integration on $\mathbb{R}^m$

Given a compact domain $D \subset \mathbb{R}^m$ and a C^0 function $f : D \to \mathbb{R}$, the integral of f over D is denoted

$$\int_D f(x)\,dx = \int_{(x^1,\ldots,x^m)\in D} f(x^1,\ldots,x^m)\,dx^1 \cdots dx^m.$$

12.2 Integral of a differential form

If Θ is a C^0 differential m-form on $O \subset \mathbb{R}^m$, then it can be written as

$$\Theta = f(x^1,\ldots,x^m)\,dx^1 \wedge \cdots \wedge dx^m,$$

where f is a C^0 function on O and $x^1,\ldots,x^m$ are the standard coordinates. The integral of Θ on a compact domain $D \subset O$ is defined to be

$$\int_D \Theta = \int_O f(x^1,\ldots,x^m)\,dx^1 \cdots dx^m, \tag{12.1}$$

Let $\mathbb{A}^m$ be an m-dimensional affine space with associated vector space $\mathbb{V}^m$ and orientation $\Omega \in \bigwedge^m(\mathbb{V}^m)^*$. If Θ is a C^0 differential m-form on an open set $O \subset \mathbb{A}^m$, then, given a diffeomorphism $\Phi : O' \to O$ such that $\Phi^*\Omega$ and $dx^1 \wedge \cdots \wedge dx^m$ determine the same orientation on $\mathbb{R}^m$, define the integral of Θ over a compact domain $C \subset O$ to be

$$\int_C \Theta = \int_{\Phi^{-1}(C)} \Phi^*\Theta,$$

where the right side is defined by (12.1).

12.3 Change of variables for the integral of a differential form

If $O, O' \subset \mathbb{R}^m$ are open, $\Phi : O \to O'$ is a C^1 diffeomorphism such that $\det d\Phi(x) > 0$, and

$$\Theta = f(y^1,\ldots,y^m)\,dy^1 \wedge \cdots \wedge dy^m$$

is a C^0 differential m-form on O', then, if $D \subset O$ is a compact domain,

$$\begin{aligned}\int_{\Phi(D)} \Theta &= \int_{\Phi(D)} f(y^1, \ldots, y^m)\, dy^1 \wedge \cdots \wedge dy^m \\ &= \int_D f(y^1(x), \ldots, y^m(x)) \left(\det \frac{\partial y}{\partial x}\right) dx^1 \wedge \cdots \wedge dx^m.\end{aligned}$$

12.4 Integration over a submanifold

If $\mathbb{A}^m$ is an m-dimensional oriented affine space, $\mathbb{B}^n$ an n-dimensional affine space, where $n \geq m$, $O \subset \mathbb{A}^m$ is open, $\Phi : O \to \mathbb{B}^n$ is a C^1 map, and Θ is a differential m-form on $\mathbb{B}^n$, then $\Phi^*\Theta$ is an m-form on O and therefore its integral over O is given by

$$\int_O \Phi^*\Theta.$$

In particular, if $\mathbb{A}^m = \mathbb{V}^m = \mathbb{R}^m$, $\mathbb{B}^n = \mathbb{W}^n = \mathbb{R}^n$, $\Phi : O \to O' \subset \mathbb{R}^n$ is a C^1 map, and Θ is a differential m-form on O' such that for each $y \in O'$,

$$\Theta(y) = \sum_{1 \leq i_1, \ldots, i_m \leq n} a_{i_1 \cdots i_m}(y)\, dy^{i_1} \wedge \cdots \wedge dy^{i_m},$$

then

$$\int_O \Phi^*\Theta = \int_O \sum_{1 \leq i_1, \ldots, i_m \leq n} a_{i_1 \cdots i_m}(y) \left(\frac{\partial y^{i_1}}{\partial x^1}\right) \cdots \left(\frac{\partial y^{i_1}}{\partial x^m}\right) dx^1 \wedge \cdots dx^m.$$

12.5 Line integral

The most basic case of integrating the pullback of a differential form is a line integral of the map is from an interval to an open set O and the differential form is a differential 1-form on O.

If $c : (a, b) \to O$ is a parameterization of an oriented curve $C \subset O$, and θ is a differential 1-form on $O \subset \mathbb{R}^m$ given by

$$\theta = f_i\, dx^i,$$

then

$$c^*\theta = f_i(c(t)) \frac{dx^i}{dt}\, dt$$

and

$$\int_C \theta = \int_{t=a}^{t=b} f_i(c(t)) \frac{dx^i}{dt}\, dt.$$

The change of variables formula above shows that the value of this integral remains the same under any reparameterization of the curve or change of coordinates on O.

Chapter 13

Exterior derivative

The normal path to Stokes's theorem is to begin by defining first differential forms, the exterior derivative, and the integral of a differential form. The path then culminates with the statement and proof of Stokes's theorem.

We take a slightly different path here. First, differential forms and the integral of a differential form are defined. We then seek a higher dimensional version of the fundamental theorem of calculus for an integral over a rectangular domain. This is accomplished using the fundamental theorem of calculus itself. The result is a simple version of Stokes's theorem. The exterior derivative of a differential form appears as the integrand of the integral over the rectangular domain. It is therefore a consequence of Stokes's theorem, rather than an a priori definition.

It is, however, necessary to show that the exterior derivative is well defined, independent of the coordinates used. This is accomplished by showing that exterior differentiation commutes with pulling back the differential form by a smooth map from the range of a smooth map to its domain.

Finally, we use the same approach to prove Stokes's theorem on a simplex. This then proves Stokes's theorem on a smoothly triangulated manifold possibly with boundary.

For convenience, $O \subset \mathbb{R}^m$ will always denote a connected open set. The domain of integration will always be a compact subset of O with piecewise smooth boundary. All functions, maps, and differential forms are assumed to be smooth.

13.1 Orientation

The space of m-forms on $\mathbb{R}^m$ is a 1-dimensional vector space and therefore the set of nonzero ones has two connected components. An orientation on $\mathbb{R}^m$ is one of the two components. Given a non-zero m-form Θ, let $[\Theta]$ denote the orientation containing Θ.

We will always use the orientation $[dx^1 \wedge \cdots \wedge dx^m]$ on $\mathbb{R}^m$.

13.2 Integral of a differential form on $\mathbb{R}^m$

Any differential m-form Θ on a domain $O \subset \mathbb{R}^m$ can written as

$$\Theta = a\,dx^1 \wedge \cdots \wedge dx^m.$$

Written in this form, the integral of Θ over the compact domain C is defined to be

$$\int_C \Theta = \int_C a\,dx^1 \cdots dx^m,$$

where the right side is an iterated integral. In particular, the integral is independent of the order of integration.

13.3 Pullback of a differential form by a smooth map

Given an open set $O' \subset \mathbb{R}^n$, a smooth map $\Phi : O \to O'$, and a differential m-form on O', the pullback of Θ by Φ is a differential m-form, denoted $\Phi^*\Theta$, on O, where, for each $x \in O$ and $v_1, \dots, v_m \in \mathbb{R}^m$,

$$(\Phi^*\Theta)(x)(v_1, \dots, v_m) = \Theta(\Phi(x))(d\Phi(x)v_1, \dots, d\Phi(x)v_m),$$

where $d\Phi$ denotes the differential of Φ.

13.4 Integral of the pullback of a differential form

Given an open set $O' \subset \mathbb{R}^n$, an embedding $\Phi : D \to O'$, tand a differential m-form Θ on O' such that Φ is orientation preservingm then the integral of Θ over the submanifold $S = \Phi(D)$ is defined to be

$$\int_S \Theta = \int_D \Phi^*\Theta,$$

The integral is independent of the parameterization Φ.

13.5 Integration over a rectangle

Recall that given an open set $O \subset \mathbb{R}^2$ and a function $f : O \to \mathbb{R}$, its differential is given by

$$df(x,y) = \partial_x f(x,y)\,dx + \partial_y f(x,y)\,dy.$$

Given $\delta, \epsilon > 0$, let

$$R = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \le x \le \delta,\ 0 \le y \le \epsilon\}.$$

We will use standard orientation on $\mathbb{R}^2$, which is given by $dx \wedge dy$, and orient the boundary of R counterclockwise. This is equivalent to saying that the orientation on each side of R is $[n\rfloor dx \wedge dy]$, where $n \in \mathbb{R}^2$ points outward.

Let $\Theta = a(x,y)\,dx + b(x,y)\,dy$ be a differential 1-form on R. Integrating it along ∂R, we get

$$\begin{aligned}
\int_{\partial R} \Theta &= \int_{x=0}^{x=\delta} a(x,0)\,dx + \int_{y=0}^{y=\epsilon} b(\delta,y)\,dy + \int_{x=\delta}^{x=0} a(x,\epsilon)\,dx + \int_{y=\epsilon}^{y=0} b(0,y)\,dy \\
&= -\int_{x=0}^{x=\delta} a(x,\epsilon) - a(x,0)\,dx + \int_{y=0}^{y=\epsilon} b(\delta,y) - b(0,y)\,dy \\
&= -\int_{x=0}^{x=\delta}\int_{y=0}^{y=\epsilon} \partial_y a(x,y)\,dy\,dx + \int_{y=0}^{y=\epsilon}\int_{x=0}^{x=\delta} \partial_x b(x,y)\,dx\,dy \\
&= \int_R (-\partial_y a(x,y) + \partial_x b(x,y))\,dx\wedge dy \\
&= \int_R (\partial_x a(x,y)\,dx + \partial_y a(x,y)\,dy)\wedge dx + (\partial_x b(x,y)\,dx + \partial_y b(x,y)\,dy)\wedge dy \\
&= \int_R da\wedge dx + db\wedge dy.
\end{aligned}$$

13.6 Integration over an m-dimensional rectangular region

Given $\delta_1,\dots,\delta_m > 0$, let

$$R = [0,\delta_1]\times\cdots\times[0,\delta_m] \subset \mathbb{R}^m.$$

For each $1\le i\le m$, let F_i denote the face that lies in the hyperplane $x^i = 0$ and $F_i + \delta_i e_i$ the one lying in the hyperplane $x^i = \delta_i$.

Let $e_1,\dots,e_m$ be the standard basis, $dx^1,\dots,dx^m$ be the dual basis, and denote $dx = dx^1\wedge\cdots dx^m$. For each $1\le i\le m$, let

$$\widehat{dx^i} = e_i \rfloor dx.$$

and therefore, $dx = dx^i\wedge\widehat{dx^i}$.

Note that, for each $1\le i\le m$, $[\widehat{dx^i}]$ is an orientation on the faces F_i and $F_i+\delta_i e_i$. On the other hand, the orientations with respect to outward pointing vectors are $[(-e_i)\rfloor dx] = -[\widehat{dx^i}]$ for F_i and $[e_i\rfloor dx] = [\widehat{dx^i}]$ for $F_i+\delta_i e_i$.

The integral of a differential $(m-1)$-form

$$\Theta = a_i(x)\widehat{dx^i}.$$

on ∂R using the outward pointing orientations is therefore

$$\begin{aligned}
\int_{\partial R} \Theta &= \sum_{i=1}^{i=m} -\int_{F_i} a_i(x)\,\widehat{dx^i} + \int_{F_i+\delta_i e_i} a_i(x)\,\widehat{dx^i} \\
&= \sum_{i=1}^{i=m} \int_{F_i} (a_i(x+\delta_i e_i) - a_i(x))\,\widehat{dx^i} \\
&= \sum_{i=1}^{i=m} \int_{F_i} \left(\int_{x^i=0}^{x^i=\delta_i} \partial_i a_i(x + x^i e_i)\, dx^i \right) \widehat{dx^i} \\
&= \sum_{i=1}^{i=m} \int_R \partial_i a_i(x)\, dx \\
&= \sum_{i=1}^{i=m} \int_R (\partial_j a_j(x)\, dx^j) \wedge \widehat{dx^i} \\
&= \int_R da_i \wedge \widehat{dx^i}.
\end{aligned}$$

This suggests that naturally associated with the differential $(m-1)$-form Θ is the differential m-form

$$d\Theta = da_i \wedge \widehat{dx^i}.$$

Using this definition, the above shows that

$$\int_{\partial R} \Theta = \int_R d\Theta.$$

13.7 The exterior derivative of a differential form

Based on the calculation above, it is reasonable to define the exterior derivative of a differential $k-1$-form

$$\Theta = \sum_{1 \le i_1, \dots, i_{k-1} \le m} a_{i_1 \cdots i_{k-1}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}},$$

to be

$$d\Theta = \sum_{1 \le i_1, \dots, i_{k-1} \le m} da_{i_1 \cdots i_{k-1}} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}},$$

It is, however, necessary to show that this definition is independent of the coordinates used. This is a consequence of the next section.

13.8 The pullback of the exterior derivative

A crucial property of the exterior derivative is that, given any differential form Θ on an open $O' \subset \mathbb{R}^n$ and a map $\Phi : O \to O'$,

$$\Phi^* d\Theta = d\Phi^*\Theta.$$

We prove this below. For convenience we denote $\Phi(x) = (y^1(x), \ldots, y^n)$.

First,

$$\begin{aligned}\Phi^*(df) &= \Phi^*\left(\frac{\partial f}{\partial y^\alpha}\, dy^\alpha\right)\\ &= \frac{\partial f}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}\, dx^i\end{aligned}$$

and, by the chain rule,

$$\begin{aligned}d(\Phi^* f) &= d(f(y(x))\\ &= \frac{\partial f}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}\, dx^i\\ &= \Phi^*(df).\end{aligned}$$

Next, if $\theta = df$, then, since partials compute,

$$\begin{aligned}d\theta &= d(df)\\ &= \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}\, dy^\alpha \wedge dy^\beta\\ &= 0\end{aligned}$$

and therefore

$$\Phi^*(d\theta) = 0.$$

On the other hand,

$$\begin{aligned}d(\Phi^*\theta) &= d(\Phi^* df)\\ &= d\left(\Phi^*\left(\frac{\partial f}{\partial y^\alpha} dy^\alpha\right)\right)\\ &= d\left(\frac{\partial f}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}\, dx^i\right)\\ &= \left(\frac{\partial^2 f}{\partial y^\beta \partial y^\alpha}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\alpha}{\partial x^i} + \frac{\partial f}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^j \partial x^i}\right) dx^j \wedge dx^i\\ &= 0.\end{aligned}$$

Finally, given a differential k−form

$$\Theta = a_{\alpha_1\cdots\alpha_k}(y)\, dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k},$$

$$\begin{aligned}\Phi^*(d\Theta) &= \Phi^*(da_{\alpha_1\cdots\alpha_k} \wedge dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k})\\ &= (\Phi^* da_{\alpha_1\cdots\alpha_k}) \wedge \Phi^*(dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k})\\ &= d(\Phi^* a_{\alpha_1\cdots\alpha_k}) \wedge \Phi^*(dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k})\\ &= d((\Phi^* a_{\alpha_1\cdots\alpha_k})\, \Phi^*(dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k}))\\ &= d(\Phi^*\Theta).\end{aligned}$$

A corollary of this is that the definition of the exterior derivative is invariant under changes of coordinates. It also shows that the exterior derivative of a differential form on a submanifold is independent of the parameterization.

13.9 Stokes's theorem for a rectangular region

Theorem 13.1. *Let $m \le n$, $R \subset \mathbb{R}^m$ be a rectangular region, and $S \subset \mathbb{R}^n$ a submanifold with a piecewise smooth boundary oriented by outward vectors, and parameterized by a map $\Phi : R \to \mathbb{R}^n$. Given any differential $(m-1)$-form Θ on an open neigborhood O' of S,*

$$\int_{\partial S} \Theta = \int_S d\Theta.$$

Proof.

$$\int_{\partial S} \Theta = \int_{\Phi(\partial R)} \Theta = \int_{\partial R} \Phi^* \Theta = \int_R d(\Phi^* \Theta) = \int_R \Phi^* (d\Theta) = \int_{\Phi(R)} d\Theta = \int_S d\Theta.$$

□

13.10 Integration over a triangle

Let $T \subset \mathbb{R}^2$ be the triangle with vertices at $(0,0), (1,0), (0,1)$. The integral of $\Theta = a\,dx + b\,dy$ along the boundary of T oriented counterclockwise is

$$\begin{aligned}
\int_{\partial T} \Theta &= \int_{x=0}^{x=1} a(x,0)\,dx + \int_{x=1}^{x=0} a(x,1-x) - b(x,1-x)\,dx + \int_{y=1}^{y=0} b(0,y)\,dy \\
&= -\int_{x=0}^{x=1} a(x,1-x) - a(x,0)\,dx + \int_{y=0}^{y=1} b(1-y,y) - b(0,y)\,dy \\
&= -\int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} \partial_y a(x,y)\,dy \right) dx + \int_{y=0}^{y=1} \left(\int_{x=0}^{x=1-y} \partial_x b(x,y)\,dx \right) dy \\
&= -\int_T \partial_y a(x,y)\,dx \wedge dy + \int_T \partial_x b(x,y)\,dx \wedge dy \\
&= \int_T (\partial_y a(x,y)\,dy + \partial_x a(x,y)\,dx) \wedge dx + (\partial_x b(x,y)\,dx + \partial_y b(x,y)\,dy) \wedge dy \\
&= \int_T da \wedge dx + db \wedge dy.
\end{aligned}$$

13.11 Integration over a simplex

Let $\Delta \subset \mathbb{R}^m$ be the simplex with vertices at $0, e_1, \ldots, e_m$. In other words,

$$\Delta = \{(x^1, \ldots, x^m) \;:\; 0 \le x^1, \ldots, x^m, x^1 + \cdots + x^m \le 1\}.$$

The boundary of Δ consists of $(n+1)$ faces, given by

$$F_0 = \{(x^1, \ldots, x^m) \;:\; x^1 + \cdots + x^m = 1,\ 0 \le x^1, \ldots, x^m \le 1\}$$

and, for each $1 \le i \le m$,

$$F_i = \{(x^1, \ldots, x^m) \;:\; x^i = 0,\ 0 \le x^1, \ldots, x^m, x^1 + \cdots + x^m \le 1\}$$

Corresponding outward vectors are $n_0 = e_1 + \cdots + e_m$ and, for each $1 \le i \le m$, $n_i = -e_i$. As before, denote $dx = dx^1 \wedge \cdots \wedge dx^m$ and

$$\widehat{dx^i} = e_i \rfloor dx.$$

The integrals below use the orientations $[\widehat{dx^1} + \cdots + \widehat{dx^m}]$ on F_0 and, for each $1 \le i \le m$, $[\widehat{dx^i}]$ on F_i. Note that, if $1 \le j \ne i \le m$, then $\widehat{dx^j}$ restricted to F_i is zero. The integral of a differential $(m-1)$-form

$$\Theta = a_j \widehat{dx^j}$$

over $\partial\Delta$ with the orientation induced by outerward vectors is therefore

$$\begin{aligned}\int_{\partial\Delta} \Theta &= \int_{F_0} a_j \widehat{dx^j} - \sum_{i=1}^{m} \int_{F_i} a_j \widehat{dx^j} \\ &= \sum_{i=1}^{m} \left(\int_{F_0} a_i \widehat{dx^i} - \int_{F_i} a_i \widehat{dx^i} \right),\end{aligned}$$

For each $1 \le i \le m$, the face F_0 can be parameterized by the map $\Phi_i : F_i \to F_0$, where

$$\Phi_i(x^1, \ldots, x^m) = (x^1, \ldots, x^m) + (1 - x^1 - \cdots - x^m) e_i.$$

It follows that

$$\begin{aligned}\int_{\partial\Delta} \Theta &= \sum_{i=1}^{m} \int_{F_i} a_i(x + (1 - x^1 - \cdots - x^m) e_i) - a_i(x) \widehat{dx^i} \\ &= \sum_{i=1}^{m} \int_{F_i} \left(\int_{x^i=0}^{x^i = 1 - x^1 - \cdots - x^m} \partial_i a_i(x + x_i e_i)\, dx^i \right) \widehat{dx^i} \\ &= \int_{\Delta} (\partial_j a_i(x)\, dx^j) \wedge \widehat{dx^i} \\ &= \int_{\Delta} da_i \wedge \widehat{dx^i}\end{aligned}$$

We therefore have proved the following theorem.

Theorem 13.2. *Let $m \le n$, $\Delta \subset \mathbb{R}^m$ be a simplex, and $S \subset \mathbb{R}^n$ a submanifold with a piecewise smooth boundary oriented by outward vectors with a parameterization $\Phi : \Delta \to \mathbb{R}^n$. Given any differential $(m-1)$-form Θ on an open neigborhood O' of S,*

$$\int_{\partial S} \Theta = \int_S d\Theta.$$

Corollary 13.3. *If Θ is a differential $(m-1)$-form on a smoothly triangulated m-dimensional manifold, then*

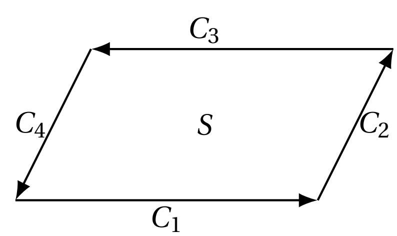
$$\int_{\partial S} \Theta = \int_S d\Theta.$$

Chapter 14

Gauss-Bonnet Theorem

14.1 Parameterizations of a polygonal surface and its boundary

Let $\overline{D} \subset \widetilde{\mathbb{R}}^2$ be a polygon with E oriented edges, labeled $\widehat{C}_1, \dots, \widehat{C}_E$, going counterclockwise around ∂D.

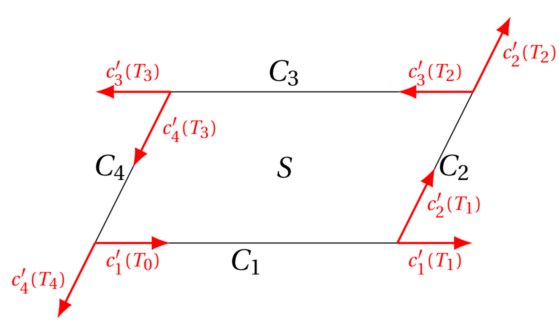


Let $\overline{S} \subset \widetilde{\mathbb{R}}^3$ be an oriented surface with boundary and a C^1 parameterization

$$\tilde{x} : \overline{D} \to \overline{S},$$

where the orientation of S is the orientation induced by $\tilde{x}$.

We can parameterize the boundary ∂S as follows:



Fix numbers $T_0 < T_1 < \cdots < T_E$. Each edge $C_k \subset \partial S$ has an oriented C^1 parameterization

$$c_k : [T_{k-1}, T_k] \to C_k,$$

where, for each $1 \le k \le E-1$,

$$c_k(T_k) = c_{k+1}(T_k) \text{ and } c_E(T_E) = c_1(T_0).$$

The vertices of $\overline{S}$ are at

$$p_1 = c_1(T_1) = c_2(T_2), \ldots, p_E = c_E(T_E) = c_1(T_0).$$

These parameterizations can be combined into a single closed curve

$$c : [T_0, T_E] \to \partial S.$$

The parameterization c_k is C^1, but the parameterization c is continuous and not necessarily C^1. At each vertex p_k, the direction of c_k need not be equal to the direction of c_{k+1}

14.2 Moving frames along a curve in a surface

Let (e_1, e_2) be a C^1 positively oriented orthonormal moving frame on $\overline{S}$. Let (ω^1, ω^2) be the dual frame, and $\omega^1_2 = -\omega^2_1$ the connection 1-form. Recall the Maurer-Cartan equations:

$$\nabla e_1 = e_2 \omega^2_1 \tag{14.1}$$

$$\nabla e_2 = e_1 \omega^1_2 \tag{14.2}$$

$$d\omega^1 + \omega^1_2 \wedge \omega^2 = 0 \tag{14.3}$$

$$d\omega^2 + \omega^2_1 \wedge \omega^1 = 0 \tag{14.4}$$

Let $C \subset \overline{S}$ be a C^1 curve with a parameterization $c : [0, T] \to C$. For convenience, given a vector field v along C, we denote

$$v'(t) = \nabla_{c'(t)} v(c(t)).$$

For each $t \in [0, T]$, let $\phi(t)$ be the angle, going counterclockwise, from $e_1(c(t))$ to $c'(t)$. Let (f_1, f_2) be a C^1 positively oriented orthonormal moving frame along C such that $f_1(t)$ is a scalar multiple of $c'(t)$, for each $t \in [0, T]$. Since (f_1, f_2) is a counterclockwise rotation of (e_1, e_2) by the angle ϕ,

$$f_1 = e_1 \cos\phi + e_2 \sin\phi$$
$$f_2 = -e_1 \sin\phi + e_2 \cos\phi.$$

Lemma 14.1.

$$\phi' = f_2 \cdot \nabla_{c'} f_1 - e_2 \cdot \nabla_{c'} e_1.$$

In particular, ϕ is C^1.

Proof.

$$\begin{aligned}
f_2 \cdot f_1' &= f_2 \cdot (e_1' \cos\phi + e_2' \sin\phi + (-e_1 \sin\phi + e_2 \cos\phi)\phi') \\
&= f_2 \cdot (e_2 \langle \omega^2_1, c' \rangle \cos\phi + e_1 \langle \omega^1_2, c' \rangle \sin\phi + f_2 \phi') \\
&= f_2 \cdot (\langle \omega^2_1, c' \rangle (e_2 \cos\phi - e_1 \sin\phi) + f_2 \phi') \\
&= \phi' + e_2 \cdot e_1'.
\end{aligned}$$

□

Define the geodesic curvature of the oriented curve C to be

$$\kappa_g = |c'|^{-1} f_2 \cdot f_1'.$$

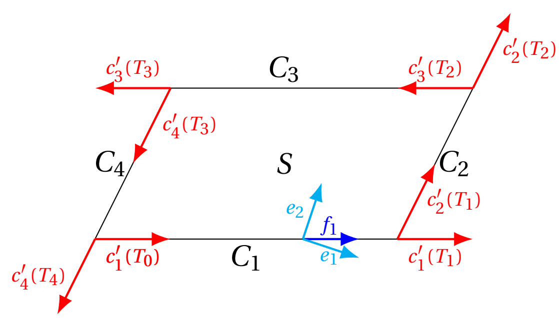
By the lemma and (14.1),

$$\begin{aligned} \phi' &= |c'|\kappa_g - \langle \omega_1^2, c'(t)\rangle \\ &= |c'|\kappa_g + \langle \omega_2^1, c'(t)\rangle. \end{aligned} \tag{14.5}$$

It follows that if we integrate the connection 1-form along C, we get

$$\begin{aligned} \int_C \omega_2^1 &= \int_{t=0}^{t=T} \langle \omega_2^1, c'(t)\rangle\, dt \\ &= \int_{t=0}^{t=T} \phi'(t) - \kappa_g |c'(t)|\, dt \\ &= \phi(T) - \phi(0) - \int_{t=0}^{t=T} \kappa_g |c'|\, dt. \end{aligned} \tag{14.6}$$

14.3 Moving frames along the boundary of a polygonal surface



Let $\overline{S}$ be a polygonal surface parameterized as described in §14.1.

For each $1 \le k \le E$ and $t \in [T_{k-1}, T_k]$, let $\phi_k(t)$ be the angle going counterclockwise from $e_1(c_k(t))$ to $c_k'(t)$.

For each $1 \le k \le E-1$, let α_k be the angle going counterclockwise from the vectors $c_k'(T_k)$ to $c_{k+1}'(T_k)$.

Let α_E is the angle going counterclockwise from $c_E'(T_E)$ to $c_1'(0)$.

Observe that, for each $1 \le k \le E-1$,

$$\phi_k(T_k) + \alpha_k = \phi_{k+1}(T_k)$$

and, when $k = E$,

$$\phi_E(T_E) + \alpha_E = \phi_1(T_0) + 2\pi,$$

because the angle has gone around the unit circle once. Therefore,

$$\begin{aligned} \phi_{k+1}(T_k) - \phi_k(T_k) &= \alpha_k \\ \phi_1(T_0) - \phi_E(T_E) &= \alpha_E - 2\pi. \end{aligned}$$

14.4 Gauss-Bonnet on a polygonal face

On one hand,

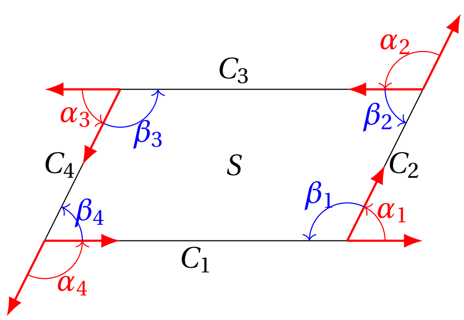
$$\begin{aligned}\int_S K\theta^1\wedge\theta^2 &= \int_S d\omega_2^1\\ &= \int_{\partial S}\omega_2^1\\ &= \sum_{k=1}^{E}\int_{C_k}\omega_2^1\\ &= \sum_{k=1}^{E}\int_{t=T_{k-1}}^{t=T_k}\langle\omega_2^1, c_k'(t)\rangle\, dt.\end{aligned}$$

On the other hand, by (14.6),

$$\begin{aligned}\sum_{k=1}^{E}\int_{t=T_{k-1}}^{t=T_k}\langle\omega_2^1, c_k'(t)\rangle\, dt &= \sum_{k=1}^{E}\phi_k(T_k)-\phi_k(T_{k-1})-\int_{t=T_{k-1}}^{t=T_k}\kappa_g|c_k'|\,dt\\ &= \phi_E(T_E)-\left(\sum_{k=2}^{E}\phi_k(T_{k-1})-\phi_{k-1}(T_{k-1})\right)\\ &\quad -\phi_1(T_0)-\sum_{k=1}^{k=E}\int_{t=T_{k-1}}^{t=T_k}\kappa_g|c_k'|\,dt\\ &= 2\pi-\alpha_1-\cdots-\alpha_E-\sum_{k=1}^{E}\int_{t=T_{k-1}}^{t=T_E}\kappa_g|c_k'|\,dt.\end{aligned}$$

Observe that each α_k is the exterior angle at each vertex. If β_k is the corresponding interior angle, then

$$\alpha_k = \pi-\beta_k.$$



Therefore,

$$\begin{aligned}\int_S K\theta^1\wedge\theta^2 &= 2\pi-\alpha_1-\cdots-\alpha_E-\sum_{k=1}^{E}\int_{t=T_{k-1}}^{t=T_k}\kappa_g|c_k'|\,dt\\ &= 2\pi-(\pi-\beta_1)-\cdots-(\pi-\beta_E)-\sum_{k=1}^{E}\int_{t=T_{k-1}}^{t=T_k}\kappa_g|c_k'|\,dt \qquad (14.7)\\ &= (2-E)\pi+\beta_1+\cdots+\beta_E-\sum_{k=1}^{E}\int_{t=T_{k-1}}^{t=T_k}\kappa_g|c_k'|\,dt.\end{aligned}$$

14.5 Gauss-Bonnet Theorem for a closed surface

Let S be a closed surface. Decompose S into F polygonal faces

$$S = S_1 \cup \cdots \cup S_F,$$

where, for each $1 \leq j \leq F$, S_j has E_j edges. Let $\beta_{j1}, \dots, \beta_{jE_j}$ denote the interior angles at each vertex of S_j. Observe that, at each vertex p in S, the sum of the interior angles at p add up to 2π. Therefore,

$$\sum_{j=1}^{F} \sum_{k=1}^{E_j} \beta_{jk} = 2\pi V,$$

where V is the number of vertices in S. Also, note that each edge lies on the boundary of two different faces but with opposite orientations. Therefore,

$$\sum_{j=1}^{F} \sum_{k=1}^{E_j} \int_{T_{j,k-1}}^{T_{j,k}} \kappa_g |c'_{j,k}|\, dt = 0.$$

Suppose each face S_j has E_j edges. Moreover, if E_j is the number of edges in the polygonal surface S_j, then

$$E_1 + \cdots + E_F = 2E,$$

where E is the number of edges in the polygonal decomposition of S. By (14.7),

$$\begin{aligned}
\int_S K\, dA &= \sum_{j=1}^{F} \int_{S_j} K\theta^1 \wedge \theta^2 \\
&= \sum_{j=1}^{F} \left((2 - E_j)\pi + \beta_{j1} + \cdots + \beta_{jE_j}) + \sum_{k=1}^{E_j} \int_{t=T_{j,k-1}}^{t=T_{j,k}} \kappa_g |c'_{j,k}(t)|\, dt \right) \\
&= 2\pi F - (E_1 + \cdots + E_F)\pi + 2\pi V \\
&= 2\pi(F - E + V).
\end{aligned}$$

This proves

Theorem 14.2 (Gauss-Bonnet Theorem)**.** *Given a C^2 closed surface $S \subset \widetilde{\mathbb{R}}^3$ and a polygonal decomposition*

$$S = S_1 \cup \cdots \cup S_F,$$

the following holds:

$$\int_S K\, dA = 2\pi(F - E + V),$$

where E is the number of edges and V is the number of vertices.

14.6 Corollaries

Corollary 14.3. *Given any two polygonal decompositions of a closed surface S with F_i faces, E_i edges, and V_i faces, where $i = 1, 2$,*

$$F_1 - E_1 + V_1 = F_2 - E_2 + V_2.$$

Corollary 14.4. *Given any two Riemannian metrics g_1 and g_2 on a closed surface S,*

$$\int_S K_1 \, dA_1 = \int_S K_2 \, dA_2,$$

where, for each $i = 1,2$, K_i is the Gauss curvature and dA_i the oriented area form of g_i.

We can therefore defined the Euler characteristic of a closed surface S to be

$$\chi(S) = F - E + V,$$

where F is the number of faces, E the number of edges, and V is the number of vertices for any polygonal decomposition of S. Equivalently, we can define it to be

$$\chi(S) = \frac{1}{2\pi} \int_S K \, dA,$$

where K is the Gauss curvature and dA is the oriented area form of any Riemannian metric on S.

Definition 14.5. Two surfaces S_1, S_2 are diffeomorphic, if there exists a bijective map $\Phi : S_1 \to S_2$ such that both Φ and Φ^{-1} are C^2.

Corollary 14.6. *If S_1 and S_2 are diffeomorphic surfaces, then $\chi(S_1) = \chi(S_2)$.*

Proof. If $\Phi : S_1 \to S_2$ is a diffeomorphism, then, given any C^1 decomposition

$$S_1 = S_{1,1} \cup \cdots \cup S_{1,F},$$

there is a corresponding C^1 decomposition

$$S_2 = S_{2,1} \cup \cdots \cup S_{2,F},$$

where $S_{2,j} = \Phi(S_{1,j})$. The two decompositions have the same number of faces, edges, and vertices. □

Chapter 15

Manifolds

A manifold is a space that up close (locally) looks like affine space but not from far away (globally). The standard examples are the sphere and torus. The difference between a manifold and a surface in affine space is that a manifold does not sit inside affine space. A surface in affine space is a manifold but not vice versa. A manifold is also the nonlinear analogue of an abstract vector space.

15.1 Category of local C^k manifolds

15.1.1 Objects

A *local C^k manifold* is an open subset of an affine space. The dimension of the local manifold is defined to be the dimension of the affine space. To indicate that a local manifold M is m-dimensional, we will often denote it as M^m and call it a *local m-manifold*.

In practice, when doing calculations, we almost always view a local m-manifold as an open subset of $\mathbb{R}^m$. This is analogous to choosing a basis for an abstract vector space.

15.1.2 Morphisms

The set of morphisms consists of all C^k maps from one local manifold to another.

In practice, when doing calculations, it is often convenient to view a C^k map $f : M^m \to N^n$ as a C^k map from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$. This is analogous to choosing bases for the domain and range of a linear map and writing the linear map as a matrix.

15.2 Category of manifolds

15.2.1 Objects

An *atlas* of a set M is a set of injective maps $f : M' \to M$, called *coordinate maps*, that satisfy the following properties:

- M' is a local manifold, often called a *coordinate chart*.

- For each $p \in M$, there exists a coordinate map $f : M' \to M$ such that $p \in f(M')$.
- If two coordinate maps $f_1 : M_1 \to M$ and $f_2 : M_2 \to M$ are such that $O = f_1(M_1) \cap f_2(M_2) \neq \emptyset$, then $f_1^{-1}(O)$ and $f_2^{-1}(O)$ are open subsets of M_1 and M_2 respectively and $f_2^{-1} \circ f_1 : f_1^{-1}(O) \to f_2^{-1}(O)$ is a C^-k diffeomorphsm (i.e., an isomorphism in the category of local C^k manifolds).

An atlas naturally induces a topology on M, where $O \subset M$ is open if and only if $f^{-1}(O) \subset M'$ is open for any coordinate map $f : M' \to M$.

Observe that the coordinate charts of each connected component of M must all have the same dimension. We will always assume that all coordinate charts of an atlas have the same dimension .

An atlas is called *locally finite*, if any $p \in M$ is contained in the images of only finitely many coordinate maps. It is *countable*, if it has countably many coordinate maps.

A set M is a C^k *manifold* with a countable locally finite atlas and the induced topology is Hausdorff. Recall that a topological space is Hausdorff, if, given any two different points $p_1, p_2 \in M$, there exist disjoint open sets O_1, O_2 such that $p_1 \in O_1$ and $p_2 \in O_2$. The dimension of M is defined to be the dimension of the coordinate charts.

Any map $f : O \to M$, where O is a local manifold, is also called a *coordinate map*, if, for any M_i such that $f(O) \cap M_i \neq \emptyset$, $f_i^{-1} \circ f : O \cap f^{-1}(f_i(M_i)) \to M_i \cap f_i^{-1}(f(O) \cap f_i(M_i))$ is a diffeomorphism.

The set of all coordinate maps for a manifold M is called the *maximal atlas* of M.

It is important to note that the definition above leaves open the possibility that a set M has two different manifold structures (i.e., two different maximal atlases) that induce the same topology. A central question in differential topology has been whether this is possible or not.

15.2.2 Morphisms

15.3 Tangent space

Given an affine space $\mathbb{A}^m$, the associated vector space V can be viewed as the space of velocity vectors. Given any C^1 curve $c : I \to \mathbb{A}^m$, $c'(t) \in V$. Conversely, given any $v \in V$, there is a C^1 curve $c : (-\delta, \delta) \to \mathbb{A}^m$ such that $c'(0) = v$. Given two points $p, q \in \mathbb{A}^m$, there is a natural affine map

$$\begin{aligned} T : \mathbb{A}^m &\to \mathbb{A}^m \\ x &\mapsto x + (q - p), \end{aligned}$$

such that for any curve c passing through p with velocity v, $T \circ c$ is a curve passing through q with the same velocity v.

On the other hand, given any C^1 map $\Phi : M \to N$, where $M \subset \mathbb{A}^m$ and $N \subset \mathbb{B}^n$, its differential defines, for each $p \in M$, a linear map

$$d\Phi(p) : V \to W.$$

Chapter 16

Riemannian manifolds

Chapter 17

Hyperbolic geometry

Appendix A

Deductive Logic

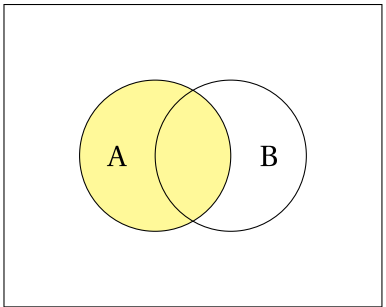
The rules of deductive logic are summarized here. These are crucial not only for proofs but also for doing correct calculations, especially those involving inequalities. It is important, for example, to know whether the order of steps in a calculation can be reversed or not. The logic underlying the use of quantifiers "for every..." and "there exists..." is also important.

A.1 Basic logical relations

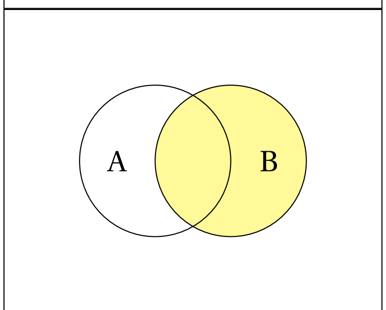
In each diagram below, A and B represent sentences that are either true (T) or false (F). Points inside the disk labeled A represent situations when A is true, and points outside the disk represent situations when A is false. Similarly, points in the disk labeled B are when B is true, and points outside the disk are when B is false.

The region shaded in yellow represents the situations where the sentence in the last column of the table is true. For example, the sentence *A and B* is true only if both A and B are true, and therefore, the only region shaded yellow in the third diagram below is in the intersection of the two disks.

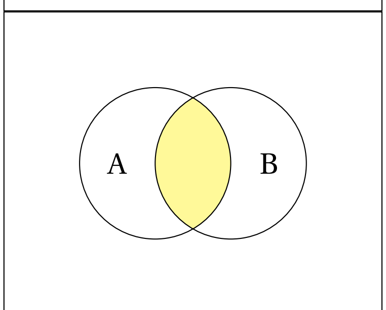
It is important when writing mathematical statements that you use the words *or* and *and* carefully. Their colloquial English meanings are not always the same as their mathematical meanings.



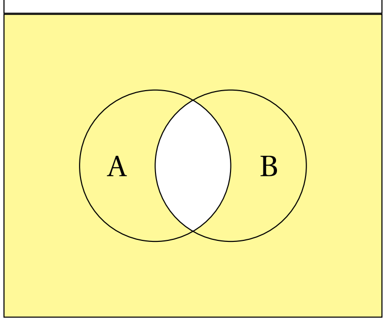
A	B	**A**
T	T	**T**
T	F	**T**
F	T	**F**
F	F	**F**



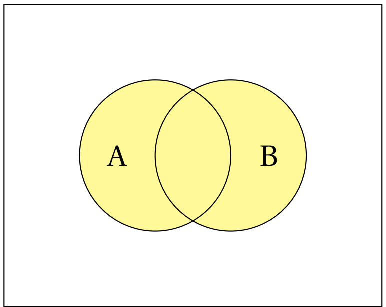
A	B	**B**
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**F**



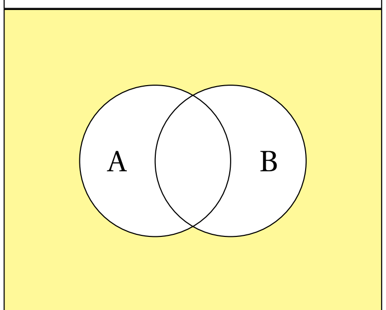
A	B	**A and B**
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**



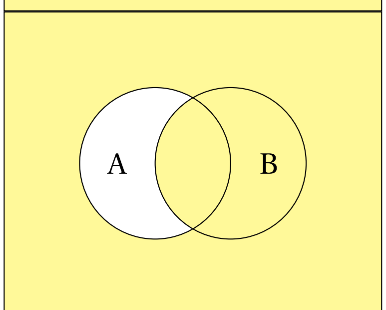
A	B	not A	not B	**(not A) or (not B)**
T	T	F	F	**F**
T	F	F	T	**T**
F	T	T	F	**T**
F	F	T	T	**T**



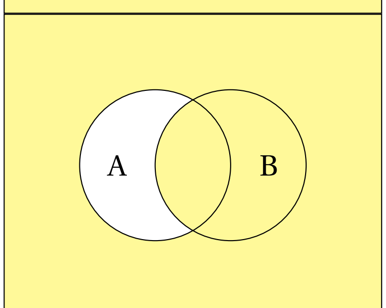
A	B	**A or B**
T	T	**T**
T	F	**T**
F	T	**T**
F	F	**F**



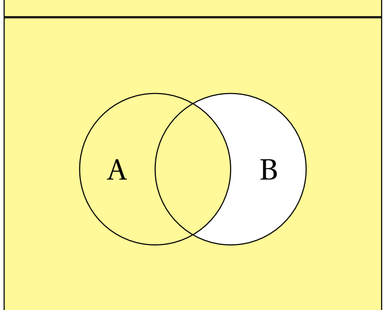
A	B	not A	not B	**(not A) and (not B)**
T	T	F	F	**F**
T	F	F	T	**F**
F	T	T	F	**F**
F	F	T	T	**T**



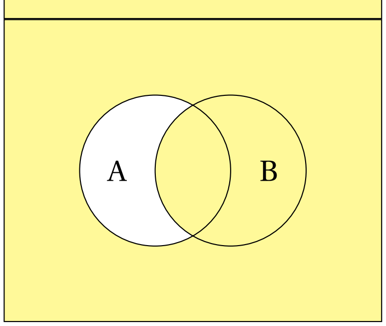
A	B	**(not A) or B**
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**



A	B	**if A, then B**
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**



A	B	**if B, then A**
T	T	**T**
T	F	**T**
F	T	**F**
F	F	**T**



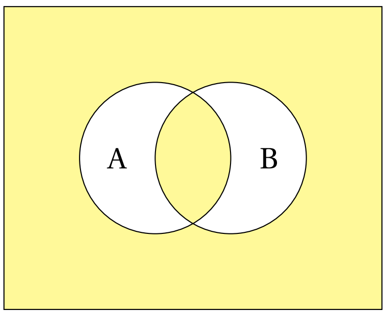
A	B	not A	not B	**if (not B) then (not A)**
T	T	F	F	**T**
T	F	F	T	**F**
F	T	T	F	**T**
F	F	T	T	**T**

A.2 Notation

$$A \implies B \text{ means if } A \text{ then } B$$
$$A \iff B \text{ means } (A \implies B) \text{ and } (B \implies A).$$

A.3 Equivalence

The diagram and table for $A \iff B$ looks like this:



A	B	A $\implies$ B	B $\implies$ A	**A $\iff$ B**
T	T	T	T	**T**
T	F	F	T	**F**
F	T	T	F	**F**
F	F	T	T	**T**

In other words, $A \iff B$ is true only if either both A and B are true or both are false. What this means is that in any mathematical sentence containing A, the A can be replaced by B without changing whether the sentence is true or false.

A.4 Negation

The diagrams above show that the following sentences are always true:

$$\text{not } (A \text{ or } B) \iff (\text{not } A) \text{ and } (\text{not } B)$$
$$\text{not } (A \text{ and } B) \iff (\text{not } A) \text{ or } (\text{not } B)$$
$$\text{not } (A \implies B) \iff A \text{ and } (\text{ not } B)$$
$$\text{not } (A \iff B) \iff (A \text{ and } (\text{not } B)) \text{ or } ((\text{not } A) \text{ and } B)$$

A.5 Converse

The converse of the statement $A \implies B$ is $B \implies A$. The converse of a statement is *not* equivalent to the statement itself. You can check this by looking at the diagrams and truth tables above.

A.6 Contrapositive

The contrapositive of the statement $A \implies B$ is

$$(\text{not } B) \implies (\text{not } A).$$

From the truth tables above, we can see that the contrapositive of a statement is equivalent to the statement itself. In other words, the following sentence is always true:

$$(A \implies B) \iff ((\text{not } A) \implies (\text{ not } B)).$$

A.7 Quantifiers

Let $A(x)$ be a statement about an element $x \in S$. The following sentences are always true:

$$(\text{ not}(\forall\, x \in S,\ A(x))) \iff (\exists\, x \in S \text{ such that not } A(x))$$
$$(\text{ not}(\exists\, x \in S,\ A(x))) \iff (\forall\, x \in S \text{ such that not } A(x)).$$

A.8 Modus Ponens

Most steps in a proof use the following process: Suppose that you have already shown that the following statements are true:

- A
- $A \implies B$

From this, you conclude that B is true. This works, because the sentence

$$(A \text{ and } (A \implies B)) \implies B$$

is always true. This can be verified using the following table:

A	B	if A, then B	A and (A $\implies$ B)	**(A and (A $\implies$ B)) $\implies$ B**
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

Note, however, that the converse is not necessarily true.

A.9 Deductive Logic

A.9.1 Direct proof

Every step in a rigorous proof must be one of the following:

- A statement is true, because it is an assumption.
- A statement is true, because it is an axiom or definition.

- A statement is true, because it has been proved elsewhere.
- A statement is true, because it has been proved earlier in this proof.
- Use modens ponens as follows:
  - Construct a new statement from other statements using the logical relations **and**, **or**, **not**, $\implies$, $\iff$.
  - Show that the statements already known to be true imply the new statement.

A.9.2 Proof by contradiction

Suppose you want to prove that a statement A is true. Assume A is false. Do a rigorous proof as above until you get a statement that is known to be false.

A.9.3 Do not start with conclusion

The first line of a proof should never be the statement you are trying to prove. You must start with a definition or an assumption and end with the statement to be proved via a sequence of logical deductions, as described above.

A.10 Notation

In the notation below, A and B represent sets and a and b represents elements.

Symbol	**Meaning**	**LaTeX**
$a \in A$	The element a lies in the set A	`a \in A`
$A \implies B$	If A, then B	`A \implies B`
$A \subset B$	A is a subset of B	`A \subset B`
$\forall\ a \in A$	For every element a in the set $A, \ldots$	`\forall\ a \in A`
$\exists\ a \in A$	There exists at least one element a in the set A,...	`\exists\ a \in A`
$a \mapsto b$	The element a maps to the element b	`a \mapsto b`
$A \rightarrow B$	A map with domain A and range B	`A \rightarrow B`

Appendix B

Vector Spaces and Linear Maps

This is a review of concepts and definitions in linear algebra, some that you should have already learned and some new. The emphasis will be on an abstract approach.

B.1 Definition of a vector space

A vector space can be defined with respect to any field $\mathscr{F}$ such as the rationals $\mathbb{Q}$, reals $\mathbb{R}$, or complex numbers $\mathbb{C}$. We, however, will restrict our attention to vector spaces with respect to the reals.

Definition B.1. A **vector space** over $\mathbb{R}$ that has the operations of addition,

$$v_1, v_2 \in \mathbb{V} \mapsto v_1 + v_2 \in \mathbb{V},$$

and scalar multiplication

$$r \in \mathbb{R},\ v \in \mathbb{V} \mapsto rv = vr \in \mathbb{V},$$

which satisfy the following properties:

$$\forall\ v_1, v_2, v_3 \in \mathbb{V},\ (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \tag{B.1}$$

$$\forall\ v_1, v_2 \in \mathbb{V},\ v_1 + v_2 = v_2 + v_1 \tag{B.2}$$

$$\exists\, \mathbf{0} \in \mathbb{V} \text{ such that } \forall\ v \in \mathbb{V},\ v + \mathbf{0} = v \tag{B.3}$$

$$\forall\ v \in \mathbb{V},\ \exists\, -v \in \mathbb{V} \text{ such that } v + (-v) = \mathbf{0} \tag{B.4}$$

$$\forall\ r_1, r_2 \in \mathbb{R} \text{ and } v \in \mathbb{V},\ (r_1 r_2) v = r_1 (r_2 v) \tag{B.5}$$

$$\forall\ r_1, r_2 \in \mathbb{R} \text{ and } v \in \mathbb{V},\ (r_1 + r_2) v = r_1 v + r_2 v \tag{B.6}$$

$$\forall\ r \in \mathbb{R} \text{ and } v_1, v_2 \in \mathbb{V},\ r(v_1 + v_2) = r v_1 + r v_2 \tag{B.7}$$

$$\forall\ v \in \mathbb{V},\ 1v = v. \tag{B.8}$$

Lemma B.2.

$$\forall\ v \in \mathbb{V},\ 0v = \mathbf{0} \tag{B.9}$$

$$\forall\ v \in \mathbb{V},\ (-1)v = -v \tag{B.10}$$

For convenience we will denote $\mathbf{0}$ by simply 0. It will always be clear, from the context, whether 0 represents the scalar 0 or the vector $\mathbf{0}$.

B.2 Examples of vector spaces

The most basic example of a vector space is $\mathbb{R}^m$. However, depending on the context, $\mathbb{R}^m$ can play different roles, sometimes as a vector space, sometimes as an affine space (defined later), and sometimes as a dual vector space. It is useful to distinguish clearly between these different types of $\mathbb{R}^m$. We therefore shall use the following notation.

Let

$$\widehat{\mathbb{R}}^m = \{\vec{v} = \langle v^1, \ldots, v^m \rangle, \text{ where } v^1, \ldots, v^m \in \mathbb{R}\}.$$

We shall also sometimes write a vector $\vec{v}$ as a column vector,

$$\vec{v} = \langle v^1, \ldots, v^m \rangle = \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix}.$$

Other examples of vector spaces are

1. The set of all solutions to a system of homogeneous linear equations.

2. Polynomials with coefficients in $\mathbb{R}$.

3. Continuous functions $f : \mathbb{R} \to \mathbb{R}$

B.3 Linearly independent vectors

Definition B.3. A finite set of vectors, $\{v_1, \ldots, v_k\} \subset \mathbb{V}$ is *linearly independent*, if, for any scalars $a^1, \ldots, a^k \in \mathbb{R}$,

$$a^1 v_1 + \cdots + a^k v_k = 0 \implies a^1 = \cdots = a^k = 0.$$

Definition B.4. Given a subset $S \subset \mathbb{V}$, the *span* of S in $\mathbb{V}$ is defined to be

$$[S] = \{a^1 v_1 + \cdots a^k v_k \; : \; \forall a^1, \ldots, a^k \in \mathbb{R}, v_1, \ldots, v_k \in S, \; k > 0\}.$$

Note that the set S need not be finite.

Lemma B.5. *Any nonempty finite set $S \subset \mathbb{V}$, where $S \neq \{0\}$, contains a linearly independent subset T such that $[T] = [S]$.*

Lemma B.6. *If $S = \{v_1, \ldots, v_k\} \subset \mathbb{V}$ is linearly independent, then, for any $v \in [S]$, there exists a unique set of scalars $a^1, \ldots, a^k \in \mathbb{R}$ such that*

$$v = a^1 v_1 + \cdots a^k v_k.$$

B.4 Basis and dimension of a vector space

Definition B.7. A vector space $\mathbb{V}$ is *finite dimensional*, if there exists a finite set $S = \{v_1, \dots, v_N\} \subset \mathbb{V}$ such that $[S] = V$.

Definition B.8. An ordered list of vectors, $E = (e_1, \dots, e_m)$, is a *basis* of $\mathbb{V}$, if the vectors are linearly independent and span $\mathbb{V}$, i.e., $[E] = \mathbb{V}$.

Lemma B.9. *Any finite dimensional vector space has at least one basis.*

Lemma B.10. *If $(e_1, \dots, e_m)$ and $(f_1, \dots, f_n)$ are both bases of $\mathbb{V}$, then $m = n$.*

Definition B.11. The dimension of a vector space $\mathbb{V}$ is m, if it has a basis with m elements. The vector space $\mathbb{V} = \{0\}$ is said to be 0-dimensional.

Lemma B.12. *If $E = (e_1, \dots, e_m)$ is a basis of $\mathbb{V}^m$, then, for each $v \in \mathbb{V}^m$, there is a unique $A = \langle a^1, \dots, a^m \rangle \in \widehat{\mathbb{R}}^m$ such that*

$$v = e_i a^i.$$

It is convenient here to write everything using formal matrix notation. First, we view a basis as a row matrix of vectors (which is why the indices are subscripts):

$$E = (e_1, \dots, e_m) = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix},$$

and an element $A \in \widehat{\mathbb{R}}^m$ as a column matrix (which is why the indices are superscripts):

$$A = \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}.$$

The vector v can now be written as

$$v = e_i a^i = EA.$$

This notation will simplify many of the formulas later.

B.5 Linear maps

A linear map from a vector space $\mathbb{V}$ to a vector space $\mathbb{W}$ (which could be $\mathbb{V}$ itself) that preserves addition and scaling. In particular, it is a map denoted

$$\begin{aligned} L : \mathbb{V} &\to \mathbb{W} \\ v &\mapsto L(v), \end{aligned}$$

where, for any $v_1, v_2, v \in \mathbb{V}$ and $a \in \mathbb{R}$,

$$\begin{aligned} L(v_1 + v_2) &= Lv_1 + Lv_2 \\ L(av) &= a(Lv). \end{aligned}$$

Therefore, given scalars $a^1, \dots, a^N \in \mathbb{R}$ and vectors $v_1, \dots, v_N$,

$$L(a^1 v_1 + \cdots + a^N v_N) = a^1 L(v_1) + \cdots + a^N L(v_N).$$

Example. Let $\mathbb{V} = \widehat{\mathbb{R}}^m$, $\mathbb{W} = \widehat{\mathbb{R}}^n$. An n-by-m matrix A defines the following linear map:

$$L : \widehat{\mathbb{R}}^m \to \widehat{\mathbb{R}}^n$$
$$\vec{v} = \langle v^1, \ldots, v^m \rangle = \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix} \mapsto A\vec{v} = \begin{bmatrix} A^1_i v^i \\ \vdots \\ A^n_i v^i \end{bmatrix} = \langle A^1_i v^i, \ldots, A^n_i v^i \rangle.$$

Conversely, given any linear map $L : \widehat{\mathbb{R}}^m \to \widehat{\mathbb{R}}^n$, there is a matrix A such that L is given by the map above.

Example. Let $\mathscr{P}$ be the set of polynomials with coefficients in $\mathbb{R}$. Differentiation defines a linear map $L : \mathscr{P} \to \mathscr{P}$, where, for each $p \in \mathscr{P}$,

$$L(p) = p'.$$

Definition B.13. A linear map $L : \mathbb{V} \to \mathbb{W}$ is *injective* or 1-1, if, for any $v_1, v_2 \in \mathbb{V}$,

$$Lv_1 = Lv_2 \iff v_1 = v_2.$$

It is *surjective* or *onto*, if, for any $w \in \mathbb{W}$, there exists $v \in \mathbb{V}$ such that $Lv = w$. It is an *isomorphism* if it is both injective and surjective. If L is an isomorphism, we denote its inverse map by L^{-1}.

Lemma B.14. *If there exists an isomorphism $L : \mathbb{V}^m \to \mathbb{W}^n$, then $m = n$.*

B.5.1 Isomorphism of an m-dimensional vector space with $\mathbb{R}^m$

Lemma B.15. *Given a basis $E = (e_1, \ldots, e_m)$ of $\mathbb{V}^m$, the map*

$$\begin{aligned} I_E : \widehat{\mathbb{R}}^m &\to \mathbb{V}^m \\ \langle v^1, \ldots, v^m \rangle &\mapsto v^1 e_1 + \cdots + v^m e_m, \end{aligned} \tag{B.11}$$

is a linear isomorphism.

B.5.2 The space of linear maps

Let $\mathrm{Hom}(\mathbb{V}^m, \mathbb{W}^n)$ denote the space of all linear maps from $\mathbb{V}^m$ to $\mathbb{W}^n$. We can define the sum of two linear maps $L_1, L_2 \in \mathrm{Hom}(\mathbb{V}^m, \mathbb{W}^n)$ to be $M : \mathbb{V}^m \to \mathbb{W}^n$, where

$$M(v) = L_1(v) + L_2(v),\ \forall\ v \in \mathbb{V}^m.$$

It is straightforward to verify that M is linear and therefore an element of $\mathrm{Hom}(\mathbb{V}^m, \mathbb{W}^n)$. We shall denote it by $L_1 + L_2$. Similar, the scalar multiple of $L \in \mathrm{Hom}(\mathbb{V}^m, \mathbb{W}^n)$ by $r \in \mathbb{R}$ is the map

$$(rL)(v) = r(L(v)),\ \forall\ v \in \mathbb{V}^m.$$

It is straightforward to show that using these definitions, $\mathrm{Hom}(Vm, \mathbb{W}^n)$ is a vector space.

Let $\mathbb{M}(n, m) = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ denote the set of n-by-m matrices.

Given a basis $E = (e_1, \dots, e_m)$ of $\mathbb{V}^m$ and a basis $F = (f_1, \dots, f_n)$ of $\mathbb{W}^n$, then for any linear map $L : \mathbb{V}^m \to \mathbb{W}^n$, there is a sequence of maps

$$\widehat{\mathbb{R}}^m \xrightarrow{I_E} \mathbb{V}^m \xrightarrow{L} \mathbb{W}^n \xrightarrow{I_F^{-1}} \widehat{\mathbb{R}}^n.$$

This defines a map

$$\begin{aligned} I_{E,F} : \operatorname{Hom}(\mathbb{V}^m, \mathbb{W}^n) &\simeq \mathbb{M}(n,m) \\ L &\mapsto I_F^{-1} \circ L \circ I_E \end{aligned}$$

Lemma B.16. *$I_{E,F}$ is a linear isomorphism, and therefore* $\operatorname{Hom}(\mathbb{V}^m, \mathbb{W}^n)$ *is an mn-dimensional vector space.*

B.5.3 The space of linear isomorphisms

Let $\operatorname{GL}(\mathbb{V})$ denote the set of all linear isomorphisms from $\mathbb{V}$ to itself. It is a subset of $\operatorname{Hom}(\mathbb{V}, \mathbb{V})$. Observe that it is *not* itself a vector space. However, it is a *group*, which means that it satisfies the following properties:

1. If $L_1, L_2 \in \operatorname{GL}(\mathbb{V})$, then $L_1 \circ \mathbb{L}_2 \in \operatorname{GL}(\mathbb{V})$.
2. If $L \in \operatorname{GL}(\mathbb{V})$, then $L^{-1} \in \operatorname{GL}(\mathbb{V})$.
3. The identity map I is an element of $\operatorname{GL}(\mathbb{V})$.

B.5.4 Subspaces and quotient spaces

Definition B.17. A set $S \subset \mathbb{V}$ is a *subspace* of $\mathbb{V}$, if it is closed under addition and scalar multiplication. As a consequence, using the same addition and scalar multiplication operations as $\mathbb{V}$, the subspace S is itself a vector space.

Example. The set

$$S = \{(x,y) \ : x + y = 0\} \subset \widehat{\mathbb{R}}^2$$

is a subspace.

Example. The set

$$T = \{(x,y) \ : \ x + y = 1\} \subset \widehat{\mathbb{R}}^2$$

is *not* a subspace. However, you can define different addition and scalar operations for which S is a vector space, namely

$$\begin{aligned} (x_1, y_1) \tilde{+} (x_2, y_2) &= (1,0) + [(x_1, y_1) - (1,0)] + [(x_2, y_2) - (1,0)] \\ &= (x_1 + x_2 - 1, y_1 + y_2) \\ a \tilde{\cdot} (x,y) &= (1,0) + a[(x,y) - (1,0)] \\ &= (1 + a(x-1), ay). \end{aligned}$$

If $S \subset \mathbb{V}$ is a linear subspace and $v \in \mathbb{V}$, let

$$v + S = \{v + s \,:\, s \in S\}.$$

If the subspace S is clearly indicated, then we can also write $[v] = v + S$. Note that

$$[v] = [w] \iff w - v \in S.$$

Definition B.18. Let

$$\mathbb{V}/S = \{v + S \,:\, v \in \mathbb{V}\}.$$

Lemma B.19. $\mathbb{V}/S$ *is a vector space, where*

$$(v + S) + (w + S) = (v + w) + S.$$

Lemma B.20. *There is a natural short exact sequence (see below) of linear maps,*

$$0 \to S \to \mathbb{V} \to \mathbb{V}/S \to 0, \tag{B.12}$$

where the

$$\begin{aligned} S &\to \mathbb{V} \\ v &\mapsto v \end{aligned}$$

and

$$\begin{aligned} \mathbb{V} &\to \mathbb{V}/S \\ v &\mapsto v + S. \end{aligned}$$

Lemma B.21. *Given a subspace* $S \subset \mathbb{V}$*,*

$$\dim \mathbb{V} = \dim S + \dim \mathbb{V}/S.$$

B.5.5 Kernel and image of a linear map

Definition B.22. Given a linear map $L : \mathbb{V} \to \mathbb{W}$, the *kernel, image,* and *cokernel* of L are defined to be

$$\begin{aligned} \ker L &= \{v \in \mathbb{V} \,:\, L(v) = 0\} \subset \mathbb{V} \\ \operatorname{im} L &= \{L(v) \,:\, v \in \mathbb{V}\} \subset \mathbb{W} \\ \operatorname{coker} L &= \mathbb{W}/\operatorname{im} L. \end{aligned}$$

Lemma B.23.

$$\begin{aligned} \ker L &\textit{ is a subspace of } \mathbb{V} \\ \operatorname{im} L &\textit{ is a subspace of } \mathbb{W} \\ \operatorname{im} L &\simeq \mathbb{V}/\ker L \\ \dim \ker L + \dim \operatorname{im} L &= \dim \mathbb{V} \\ \dim \operatorname{im} L + \dim \operatorname{coker} L &= \dim \mathbb{W}. \end{aligned}$$

Definition B.24. The *rank* of L is defined to be

$$\operatorname{rank} L = \dim \operatorname{im} L.$$

Therefore, the identities above can also be written as

$$\dim \ker L + \operatorname{rank} L = \dim \mathbb{V}$$
$$\operatorname{rank} L + \dim \operatorname{coker} L = \dim \mathbb{W}.$$

B.5.6 Exact sequence of linear maps

Definition B.25. A chain of two linear maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact*, if $\operatorname{im} f = \ker g$. A longer chain of maps is exact, if any two consecutive maps are exact.

The exactness of the sequence (B.12) means that the inclusion map $S \subset \mathbb{V}$ is injective, the kernel of the map $\mathbb{V} \to \mathbb{V}/S$ is S, and the map $\mathbb{V} \to \mathbb{V}/S$ is surjective.

B.6 Normal forms of a linear transformation

Lemma B.26. *Let $L : \mathbb{V} \to \mathbb{W}$ be a linear map. There exists a basis $E = (e_1, \dots, e_m)$ of $\mathbb{V}$ and a basis F of $\mathbb{W}$ such that the matrix $M = I_{E,F}(L)$ satisfies the following:*

$$M^i_j = \begin{cases} \delta^i_j & \text{if } 1 \le i, j \le k \\ 0 & \text{if } k+1 \le i \le n \text{ and } 1 \le j \le m, \end{cases} \tag{B.13}$$

where $k = \operatorname{rank} L$.

Proof. Let $\dim \ker L = m - k$. There exists a basis $(e_{k+1}, \dots, e_m)$ of $\ker L$, which can be extended to a basis $E = (e_1, \dots, e_m)$ of $\mathbb{V}$. The vectors $f_1 = L(e_1), \dots, f_k = L(e_k)$ are linearly independent, and therefore can be extended to a basis $F = (f_1, \dots, f_n)$. The matrix $M = I_{E,F}(L)$ satisfies (B.13). □

B.7 Change of basis

Let E and F be bases of a vector space $\mathbb{V}^m$. Therefore, there exists a matrix

$$R = \begin{bmatrix} R_{11} & \cdots & R_{1m} \\ \vdots & & \vdots \\ R_{m1} & \cdots & R_{mm} \end{bmatrix}$$

such that, for each $1 \le j \le m$,

$$f_j = e_i R^i_j.$$

Equivalently, if E and F are treated as row matrices of vectors, then

$$F = ER.$$

Using this notation, it is easy to derive the change of basis formulas for a vector. For each $v \in \mathbb{V}^m$, there exist coefficients $A, B \in \widehat{\mathbb{R}}^m$, viewed as column vectors, such that

$$v = EA = FB = FRB.$$

Since the coefficient matrices A and B are uniquely determined by the bases and the vector v, it follows that

$$A = RB.$$

B.8 Dual vector space

Definition B.27. Given a vector space $\mathbb{V}$, its *dual vector space* $\mathbb{V}^*$ is the set of all linear functions of $\mathbb{V}$,

$$\mathbb{V}^* = \{\theta : V \to \mathbb{R} \,:\, \theta \text{ is linear}\}.$$

For convenience, we shall use the following angle bracket notation: Given $\theta \in \mathbb{V}^*$ and $v \in \mathbb{V}$,

$$\langle \theta, v\rangle = \langle v, \theta\rangle = \theta(v).$$

Lemma B.28. *Given any linear map $L : \mathbb{V} \to \mathbb{W}$, there is a unique linear map $L^t : \mathbb{W}^* \to \mathbb{V}^*$, such that, for any $v \in \mathbb{V}$ and $\eta \in \mathbb{W}^*$,*

$$\langle \eta, L(v)\rangle = \langle L^*(\eta), v\rangle.$$

Lemma B.29. *There is a natural isomorphism* $(\mathbb{V}^*)^* = \mathbb{V}$.

Given a basis $(e_1, \ldots, e_m)$ of $\mathbb{V}$, there is a natural basis $(\theta^1, \ldots, \theta^m)$ of $\mathbb{V}^*$, where, for each $1 \le i, j \le m$,

$$\langle \theta^i, e_j\rangle = \delta^i_j.$$

In particular, if $v = a^i e_i$, then, for each $1 \le j \le m$,

$$\langle \theta^j, v\rangle = a^j.$$

B.9 Dual Linear Map

Given any linear map $L : \mathbb{V} \to \mathbb{W}$, there is a naturally induced *dual map* $L^* : \mathbb{W}^* \to \mathbb{V}^*$ defined as follows: For any $\omega \in \mathbb{W}^*$, $L^*\omega \in \mathbb{V}^*$ is defined as follows: For any $v \in V$,

$$\langle L^*\omega, v\rangle = \langle \omega, L(v)\rangle.$$

Lemma B.30. *The map*

$$\begin{aligned} \mathrm{Hom}(\mathbb{V}, \mathbb{W}) &\to \mathrm{Hom}(\mathbb{W}^*, \mathbb{V}^*) \\ L &\mapsto L^* \end{aligned}$$

is a linear isomorphism.

B.10 Duality between subspace and quotient space

Given a linear subspace $S \subset \mathbb{V}$, define its *annihilator* to be

$$S^\perp = \{\theta \in \mathbb{V}^* \;:\; \langle \theta, v \rangle = 0,\ \forall\, v \in S\}.$$

Lemma B.31. *Given a subspace $S \subset \mathbb{V}$, there are natural linear isomorphisms*

$$S^* = \mathbb{V}^* / S^\perp$$

$$(V/S)^* = S^\perp$$

Appendix C

Tensors

C.1 Multilinear functions

Let $\mathbb{V}^m$ be an m-dimensional vector space. Recall that a function $\ell : \mathbb{V}^m \to \mathbb{R}$ is linear, if for any $v, w \in \mathbb{V}^m$ and $a, b \in \mathbb{R}$,

$$\ell(av + bw) = a\ell(v) + b\ell(w)m$$

and the set of all linear functions is a vector space denoted $(\mathbb{V}^m)^*$.

A k-linear function is a function with k inputs such that if all but one inputs are held fixed, then it is a linear function of the remaining input. In particular, a function

$$f : \mathbb{V}^m \times \cdots \times \mathbb{V}^m \to \mathbb{R},$$

is k-linear, if for any $1 \le i \le k$, vectors $v_1, \dots, v_i, \dots, v_k, w \in \mathbb{V}^m$, and $a, b \in \mathbb{R}$,

$$f(v_1, \dots, av_i + bw, \dots, v_k) = af(v_1, \dots, v_i, \dots, v_k) + bf(v_1, \dots, w, \dots, v_k).$$

A k-linear function is also called a k-tensor. A multilinear function or tensor is a k-linear function for some k.

The set of all k-tensors on $\mathbb{V}^m$ is itself a vector space, which we denote by

$$(\mathbb{V}^m)^* \otimes \cdots \otimes (\mathbb{V}^m)^* = \otimes^k (\mathbb{V}^m)^*.$$

Given a basis $e_1, \dots e_m \in \mathbb{V}^m$, each $f \in \otimes^k(\mathbb{V}^m)^*$ is uniquely determined by its values $f(e_{i_1}, \dots, e_{i_k})$, $1 \le i_1, \dots, i_k \le m$, and vice versa. Therefore, $\otimes^k(\mathbb{V}^m)^*$ is an m^k-dimensional vector space.

The basic examples are the following:

- A 1-tensor on $\mathbb{V}$ is an element of $\mathbb{V}^*$.
- The dot product on $\mathbb{R}^m$ is a 2-tensor.

C.2 Symmetric tensors

A k-tensor f is *symmetric*, if the value of f remains unchanged when the inputs are permuted. Specifically, f is symmetric, if, for any $1 \le i < j \le k$, $v_1, \dots, v_i, \dots, v_j, \dots, v_k$,

$$f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = f(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

Since such transpositions generate the entire group S_k of permutations of k elemnts, it follows that for any $v_1, \ldots, v_k \in \mathbb{V}^m$ and $\sigma \in S_k$,

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k).$$

For example, a second order tensor f is symmetric if, for any $v, w \in \mathbb{V}^m$,

$$f(v, w) = f(w, v),$$

and a third order tensor f is symmetric if, for any $u, v, w \in \mathbb{V}^m$,

$$f(u, v, w) = f(v, u, w) = f(v, w, u) = f(w, v, u) = f(w, u, v) = f(w, v, u).$$

The set of all symmetric k-order tensors is a subspace of $\otimes^k(\mathbb{V}^m)^*$, denoted $S^k(\mathbb{V}^m)^*$. Since a symmetric k-tensor f is uniquely determined by its values $f(e_{i_1}, \ldots, e_{i_k})$ for all $1 \le i_1 \le i_2 \le \cdots \le i_k \le m$, its dimension is equal to the set of all ordered partitions of $\{1, \ldots, m\}$, which is

$$\dim S^k(\mathbb{V}^m)^* = \binom{m+k-1}{k}.$$

Example. The dot product on $\widehat{\mathbb{R}}^m$ is a symmetric 2-tensor.

C.3 Antisymmetric tensors

A k-tensor f is *antisymmetric* or *alternating* or *exterior*, if, when any two inputs to f as swapped, the new value of f is equal to minus the original value. Specifically, f is symmetric, if, for any $1 \le i < j \le k$, $v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k$,

$$f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k). \tag{C.1}$$

This can be written in terms of permutations as follows. Given any permutation $\sigma \in S_k$, we can define its parity, denoted $(-1)^\sigma$ as follows: If

$$\sigma = \tau_1 \cdots \tau_p,$$

then

$$(-1)^\sigma = \begin{cases} 1 & \text{if } p \text{ is even} \\ -1 & \text{if } p \text{ is odd.} \end{cases}$$

A tensor f is antisymmetric, if for any $v_1, \ldots, v_k \in \mathbb{V}^m$ and $\sigma \in S_k$,

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (-1)^\sigma f(v_1, \ldots, v_k).$$

An immediate consequence of (C.1) is that if any two inputs to f are the same, then the value of f is zero.

The set of all antisymmetric k-tensors is a subspace of $\otimes^k(\mathbb{V}^m)^*$, denoted $\bigwedge^k(\mathbb{V}^m)^*$. Since an antisymmetric tensor f is uniquely determined by its values $f(e_{i_1}, \ldots, e_{i_k})$, where $1 \le i_1 < i_2 \cdots < i_k \le k$, and vice versa,

$$\dim \bigwedge^k(\mathbb{V}^m)^* = \binom{m}{k}.$$

An exterior k-tensor is also called a k-form.

The cross product on $\widehat{\mathbb{R}}^2$, defined to be

$$\langle v_1, v_2\rangle \times \langle w_1, w_2\rangle = v_1 w_2 - v_2 w_1,$$

is an antisymmetric 2-tensor. The function

$$(u, v, w) \mapsto u \cdot (v \times w), \text{ where } u, v, w \in \widehat{\mathbb{R}}^3,$$

is an antisymmetric 3-tensor on $\widehat{\mathbb{R}}^3$. More generally, the function

$$D(v_1, \dots, v_m) = \det \left[v_1 | \cdots | v_m \right]$$

is an anti-symmetric m-tensor on $\widehat{\mathbb{R}}^m$.

C.4 The pullback of a tensor

Given a linear map $L : \mathbb{V} \to \mathbb{W}$ and a dual vector $\omega \in \mathbb{W}^*$, the *pullback* of ω by the map L is defined to be $L^t(\omega^*)$, where L^t is the transpose of L, as defined in §B.9. This can be generalized to pulling back tensors.

If $f \in \otimes^k(\mathbb{W}^n)^*$ and $L : \mathbb{V}^m \to \mathbb{W}^n$ is a linear map, then the *pullback* of f by L is the k-tensor, denoted $L^* f \in \otimes^k(\mathbb{V}^m)^*$, defined by

$$L^* f(v_1, \dots, v_k) = f(Lv_1, \dots, Lv_k).$$

Since $(a_1 L_1 + a_2 L_2)^* f = a_1 L_1^* f + a_2 L_2^* f$, for any linear maps $L_1, L_2 : \mathbb{V}^m \to \mathbb{W}^n$, $a_1, a_2 \in \mathbb{R}$, and $f \in \otimes^k(\mathbb{V}^m)^*$, the pullback of L is a linear map

$$L^* : \otimes^k(\mathbb{W}^n)^* \to \otimes^k(\mathbb{V}^m)^*.$$

If $f \in S^k(\mathbb{W}^n)^*$, then $L^* f \in S^k(\mathbb{V}^m)^*$, and, if $f \in \bigwedge^k(\mathbb{W}^n)^*$, then $L^* f \in \bigwedge^k(\mathbb{V}^m)^*$.

Note that, for 1-tensors, $L^* = L^t$.

C.5 The determinant of a linear map

If $\dim \mathbb{V}^m = m$, then

$$\dim \bigwedge^m (\mathbb{V}^m)^* = 1.$$

It follows that if $L : \mathbb{V}^m \to \mathbb{V}^m$ is a linear map, then there exists $c \in \mathbb{R}$ such that the pullback map $L^* : \bigwedge^m(\mathbb{V}^m)^* \to \bigwedge^m(\mathbb{V}^m)^*$ is given by

$$L^* f = cf,$$

for any $f \in \bigwedge^k(\mathbb{V}^m)^*$. The scalar c is defined to be the *determinant* of L and denoted $\det L$. In other words, for any $f \in \bigwedge^m(\mathbb{V}^m)^*$,

$$L^* f = (\det L) f.$$

If L is the identity map, then $L^* f = f$ and therefore, $\det L = 1$.

C.6 Orientation

Since $\dim \bigwedge^m(\mathbb{V}^m)^* = 1$, $\bigwedge^m(\mathbb{V}^m)^* \setminus \{0\}$ has two connected components. Given $\Theta \in \bigwedge^m(\mathbb{V}^m)^* \setminus \{0\}$, let

$$[\Theta] = \{t\Theta \,:\, t \in \mathbb{R}^+\}$$

be the component that contains Θ and

$$(\bigwedge^m(\mathbb{V}^m)^* \setminus \{0\})/\mathbb{R}^+ = \{[\Theta] \,:\, \Theta \in \bigwedge^m(\mathbb{V}^m)^* \setminus \{0\}\}.$$

An orientation of $\mathbb{V}^m$ is an element $[\Theta] \in (\bigwedge^m(\mathbb{V}^m)^* \setminus \{0\})/\mathbb{R}^+$.

Any basis $E = (e_1, \dots, e_m)$ of $\mathbb{V}^m$ determines an orientation $[\omega^1 \wedge \cdots \wedge \omega^m]$, where $\omega^1, \dots, \omega^m$ is the dual basis. Note that, if $\sigma \in S_m$, then

$$[\omega^{\sigma(1)} \wedge \cdots \wedge \omega^{\sigma(m)}] = (-1)^\sigma [\omega^{\sigma(1)} \wedge \cdots \wedge \omega^{\sigma(m)}].$$

C.7 Exterior product

If $\theta^1, \theta^2 \in (\mathbb{V}^m)^* = \bigwedge^1(\mathbb{V}^m)^*$, then we can define a 2-form, denoted $\theta^1 \wedge \theta^2$, as follows: For any $v_1, v_2 \in \mathbb{V}^m$,

$$(\theta^1 \wedge \theta^2)(v_1, v_2) = \theta^1(v_1)\theta^2(v_2) - \theta^1(v_2)\theta^2(v_1).$$

More generally, define the *wedge product* of a k-form α and an l-form β to be the $(k+l)$-form $\alpha \wedge \beta$ given by

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+1}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}).$$

In particular, if $\theta^1, \dots, \theta^k$, then

$$(\theta^1 \wedge \cdots \wedge \theta^k)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} (-1)^\sigma \langle \theta^1, v_{\sigma(1)} \rangle \cdots \langle \theta^k, v_{\sigma(k)} \rangle$$

The constant in the definition is chosen so that the following holds: If $e_1, \dots, e_m$ is a basis of $\mathbb{V}^m$, $\omega^1, \dots, \omega^m \in (\mathbb{V}^m)^*$ the dual basis, then

$$(\omega^1 \wedge \cdots \omega^k)(e_1, \dots, e_k) = 1.$$

C.8 Interior product

If $v \in \mathbb{V}^m$, there is a linear map denoted

$$\begin{aligned} \bigwedge^{k+1}(\mathbb{V}^m)^* &\to \bigwedge^{k}(\mathbb{V}^m)^* \\ \theta &\mapsto v \rfloor \theta, \end{aligned}$$

where

$$(v \rfloor \theta)(v_1, \dots, v_{k-1}) = \theta(v, v_1, \dots, v_k).$$

Appendix D

Calculus on $\mathbb{R}^m$

Throughout this chapter, O is an open subset of $\mathbb{R}^m$. See Definition D.1 for the definition of an open set.

D.1 Topology of $\mathbb{R}^m$

Given $p_0 \in \mathbb{R}^m$ and $r > 0$, the open ball of radius r centered at p_0 is denoted

$$B(p_0, r) = \{p \in \mathbb{R}^m : |p - p_0| < r\}.$$

Definition D.1. A set $O \subset \mathbb{R}^m$ is *open*, if and only if, for each $p_0 \in O$, there exists $r > 0$ such that $B(p_0, r) \subset O$.

D.2 Continuity

If $O \subset \mathbb{A}^m$ is open, a map $f : O \to \mathbb{B}^n$ is *continuous*, if, for any open set $O' \subset \mathbb{B}^n$, $f^{-1}(O') \subset \mathbb{A}^m$ is open.

Lemma D.2. *If $O \subset \mathbb{R}^m$ is open, a function $f : O \to \mathbb{R}$ is continuous if and only if the following holds: For any $p_0 \in O$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|p - p_0| < \delta \implies |f(p) - f(p_0)| < \epsilon.$$

Lemma D.3. *A function $f : O \subset \mathbb{R}$ is continuous if and only if for any open interval $I \subset \mathbb{R}$, $f^{-1}(I) \subset O$ is open.*

D.3 Derivatives

D.3.1 Partial derivatives

For each $1 \le i \le m$, we denote the partial derivative of a function $f : O \to \mathbb{R}$ with respect to the i-th input variable by $\partial_i f$. Higher order partial derivatives will be denoted $\partial^\alpha f$, where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_i \ge 0$, $1 \le i \le m$, and

$$\partial_\alpha = (\partial_1)^{\alpha_1} \cdots (\partial_m)^{\alpha_m} f.$$

The order of ∂^α is defined to be

$$|\alpha| = \alpha_1 + \cdots + \alpha_m.$$

If the partials of f up to order k exist and are continuous, we say that f is C^k.

D.3.2 Directional derivative

If f is C^1, then given $p \in O$, and $v \in \mathbb{R}^m$, then the directional derivative of f at p in the direction v is defined to be

$$d_v f(p) = \frac{d}{dt}\bigg|_{t=0} f(p + tv). \tag{D.1}$$

Recall that

$$d_v f(p) = v^i \partial_i f(p). \tag{D.2}$$

Lemma D.4. *If $\delta > 0$ and $c : (-\delta, \delta) \to O$ is a C^1 curve such that $c(0) = p$ and $c'(0) = v$, then*

$$d_v f(p) = \frac{d}{dt}\bigg|_{t=0} f(c(t)).$$

D.3.3 Differential of a function

Given a C^1 function $f : O \to \mathbb{R}$ and $p \in O$, define the function

$$\begin{aligned} df(p) : \mathbb{R}^m &\to \mathbb{R} \\ v &\mapsto d_v f(p) \end{aligned}$$

By (D.2), $df(p)$ is linear and therefore $df(p) \in (\mathbb{R}^m)^*$. Therefore, for each $v \in \mathbb{R}^m$,

$$d_v f(p) = \langle v, df(p) \rangle.$$

The map $df : O \to (\mathbb{R}^m)^*$ is called the *differential of f*. We will also denote the differential of f by ∂f, depending on the context.

D.3.4 Hessian of a function

If $f : O \to \mathbb{R}$ is C^2, then the matrix of second order partials,

$$Hf(p) = \begin{bmatrix} H_{11} & \cdots & H_{1m} \\ \vdots & & \vdots \\ H_{m1} & \cdots & H_{mm} \end{bmatrix},$$

where, for each $1 \le i, j \le m$,

$$H_{ij}(p) = \partial^2_{ij} f(p),$$

is called the *Hessian* of f at p.

Recall that if f is C^2, then for any $1 \le i, j \le m$,

$$\partial^2_{ij} f = \partial^2_{ji} f.$$

This implies that the Hessian $H(p)$ is a symmetrix matrix.

D.3.5 Jacobian of a map

A map $f = (f^1, \ldots, f^n) : O \to \mathbb{R}^n$ is C^k if, for each $1 \le a \le n$, the function $f^a : O \to \mathbb{R}$ is C^k. The matrix of partial derivatives

$$\partial f = \begin{bmatrix} \partial_1 f & \cdots & \partial_m f \end{bmatrix} = \begin{bmatrix} \partial_1 f^1 & \cdots & \partial_m f^1 \\ \vdots & & \vdots \\ \partial_1 f^n & \cdots & \partial_m f^n \end{bmatrix}$$

is called the *Jacobian* of f at p.

D.3.6 Chain Rule

Lemma D.5. *If $O \subset \mathbb{R}^m$ and $O' \subset \mathbb{R}^n$ are open, $F : O \to O'$ is a C^1 map, and $G : O' \to \mathbb{R}$ is a C^1 map, then $G \circ F : O \to \mathbb{R}$ is C^1 and*

$$d(G \circ F)(p) = dG(F(p))dF(p),$$

where the right is matrix mulitiplication of $dG(F(p))$ b $df(p)$.

D.3.7 Differentiation on region with boundary

Let

$$\mathbb{H}^m = \{(x^1, \ldots, x^m) \in \mathbb{R}^m \ : \ x^m \le 0\}.$$

and denote its boundary by

$$\partial\mathbb{H}^m \{(x^1, \ldots, x^m) \in \mathbb{R}^m \ : \ x^m = 0\}.$$

Given an open set $O \subset \mathbb{R}^m$ and a function $f : O \cap \mathbb{H}^m \to \mathbb{R}$, the partial derivatives $\partial_i f$, where $1 \le i \le m-1$ can be defined as usual. If $x \in O \cap \partial\mathbb{H}^m = \emptyset$, then the usual definition of $\partial_m f(x)$ works. If, however, $x \in \partial\mathbb{H}^m$, a one-sided limit is needed:

$$\frac{\partial f}{\partial x^m}(x) = \lim_{h \to 0-} \frac{f(x^1, \ldots, x^{m-1}, x^m + h)}{h}.$$

Given that, the definition of a C^k function or map on $O \cap \mathbb{H}^m$ is the same as before.

D.4 Integration

D.4.1 Fundamental Theorem of Calculus

If $f : [a, b] \to \mathbb{R}$ is C^1, then

$$\int_{x=a}^{x=b} f'(x)\, dx = f(b) - f(a).$$

D.4.2 Integral over a rectangular region

Consider a closed rectangular region

$$R = [a^1, b^1] \times \cdots [a^m, b^m] = \{(x^1, \ldots, x^m) \,:\, a^i \le x^i \le b^i,\ 1 \le i \le m\} \subset O \subset \mathbb{R}^m.$$

The integral over R of a continuous function $f : O \to \mathbb{R}$ is defined to be

$$\int_R f(x)\,dx = \int_{x^m=a^m}^{x^m=b^m} \cdots \int_{x^m=a^m}^{x^m=b^m} f(x^1, \ldots, x^m)\,dx^1 \cdots dx^m.$$

By the Fubini theorem, it does not matter what order the integrals are carried out.

D.4.3 Orientation

An ordered pair of vectors (v, w) in 2-space has **positive orientation**, if w lies between v and $-v$ going **counterclockwise**.

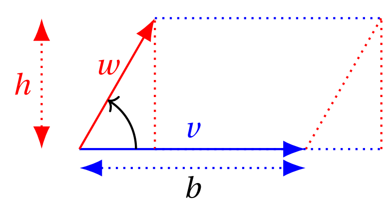
D.4.4 Oriented area of parallelogram

Consider the parallelogram with a vertex at the origin and the two sides touching the origin given by vectors v, w. Recall that the area of this parallelogram is base times height.

Let $A(v, w)$ denote the oriented area of the parallelogram with respect to (v, w). If (v, w) is positively oriented, then $A(v, w)$ is the area, if (v, w) is negatively oriented, then $A(v, w)$ is equal to minus the area. Observe that

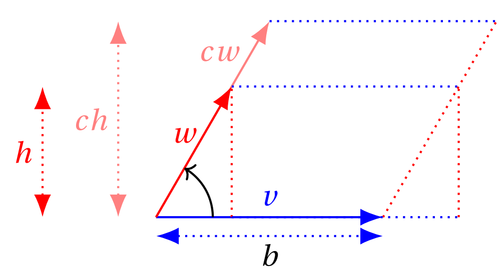
$$A(w, v) = -A(v, w), \tag{D.3}$$

because the orientation of (w, v) is minus the orientation of (v, w).



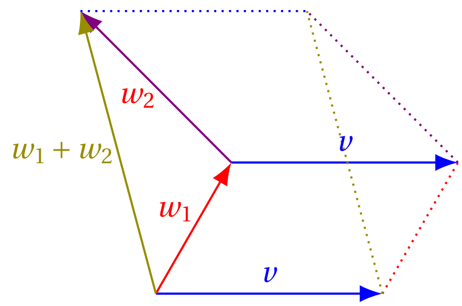
If one side is rescaled by a factor of c, the oriented area is rescaled by c,

$$A(v, cw) = cA(v, w). \tag{D.4}$$



Observe that

$$A(v, w_1 + w_2) = A(v, w_1)A(v, w_2) \tag{D.5}$$



By (D.3), (D.4), and (D.5), if $v = (v^1, v^2)$, $w = (w^1, w^2)$, and (e_1, e_2) is the standard basis of $\mathbb{R}^2$, then

$$\begin{aligned} A(v,w) &= A(v^1 e_1 + v^2 e_2, w^1 e_1 + w^2 e_2) \\ &= v^1 w^1 A(e_1,e_1) + v^1 w^2 A(e_1,e_2) + v^2 w^1 A(e_2,e_1) + v^2 w^2 A(e_2,e_2) \\ &= (v^1 w^2 - v^2 w^1) A(e_1,e_2) \\ &= v^1 w^2 - v^2 w^1. \end{aligned}$$

D.4.5 Oriented volume of a parallelotope in $\mathbb{R}^m$

The parallelotope spanned by a basis $(v_1,\dots,v_m)$ of $\mathbb{R}^m$ is defined to be

$$P(v_1,\dots,v_m) = \{a^1 v_1 + \cdots + a^m v_m \,:\, 0 \le a^1,\dots,a^m \le 1\}.$$

The oriented volume of $P(v_1,\dots,v_m)$, denoted $V(v_1,\dots,v_m)$, is defined to be the unique anti-symmetric multilinear function such that

$$V(e_1,\dots,e_m) = 1,$$

where $(e_1,\dots,e_m)$ is the standard basis of $\mathbb{R}^m$. In particular, if $\sigma = (\sigma_1,\dots,\sigma_m)$ is a permutation of $1,\dots,m$ and $(-1)^\sigma$ is the parity of the permutation, then

$$V(e_{\sigma_1},\dots,e_{\sigma_m}) = (-1)^\sigma V(e_1,\dots,e_m).$$

This implies that the volume is given by

$$V(v_1,\dots,v_m) = \det \begin{bmatrix} v_1^1 & \cdots & v_m^1 \\ \vdots & & \vdots \\ v_1^m & \cdots & v_m^m \end{bmatrix}.$$

D.4.6 Parameterization of parallelogram by a square

The parallelogram $P(v_1, v_2)$ can be linearly parameterized by the unit square $S = [0,1] \times [0,1]$ by the linear map

$$L(t^1, t^2) = t^1 v_1 + t^2 v_2,\ (t^1, t^2) \in [0,1] \times [0,1].$$

D.4.7 Integral over region parameterized by rectangular region

Let $D \subset \mathbb{R}^2$ be parameterized by a map

$$\Phi : R \to D,$$

where $R = [0, S] \times [0, T]$. Suppose we chop R into small squares with sides of length $\delta > 0$. Suppose x lies in the center of the small square

$$r(x,\delta) = \left\{ x + \delta(t^1, t^2) \; : \; -\frac{1}{2} \le t^1, t^2 \le \frac{1}{2} \right\}.$$

The image of $r(x)$ under the map Φ is approximately the parallelogram

$$P = \left\{ \Phi(x) + t^1 v_1 + t^2 v_2 \; : \; -\frac{1}{2} \le t^1, t^2 \le \frac{1}{2} \right\},$$

where $v_1 = \delta \partial_1 \Phi(x)$ and $v_2 = \delta \partial_2 \Phi(x)$. Therefore, the oriented area of $r(x,\delta)$ is approximately

$$a(x,\delta) \simeq |\det \partial \Phi(x)| \delta^2.$$

Therefore,

$$\begin{aligned} \int_D f(y)\, dy &\simeq \sum_x f(\Phi(x))\, a(x,\delta) \\ &\simeq \delta^2 \sum_x f(\Phi(x)) |\det(\partial \Phi(x))| \\ &\simeq \int_R f(\Phi(x)) |\det(\partial \Phi(x))|\, dx \end{aligned}$$

Taking a limit, we get

$$\int_D f(y), dy = \int_R f(\Phi(x)) \, |\det(\partial \Phi(x))|\, dx.$$

This extends to higher dimensions: Given a region $D \subset \mathbb{R}^m$ and a parameterization $\Phi : R \to D$,

$$\int_D f(y), dy = \int_R f(\Phi(x)) |\det(\partial \Phi(x))|\, dx.$$

Appendix E

Normal forms of maps

E.1 Normal form of a linear map

Lemma E.1. *Let $L : \mathbb{V}^m \to \mathbb{W}^n$ be a linear map. There exist invertible linear maps $A : \mathbb{R}^m \to \mathbb{V}^m$ and $B : \mathbb{R}^n \to \mathbb{W}^n$ such that*

$$B^{-1}LA = \left[\begin{array}{c|c} I_r & 0_{r,m-r} \\ \hline 0_{n-r,r} & 0_{n-r,m-r} \end{array}\right],$$

where I_r is the r-by-r identity matrix and $0_{k,l}$ is the k-by-l zero matrix.

It follows that the maximum possible rank of a linear map $L : \mathbb{V}^m \to Wn$ is $\min(m, n)$.

If $m \geq n$ and $L : \mathbb{V}^m \to \mathbb{W}^n$ has maximum rank, then the rank is n,

$$L = B \begin{bmatrix} I_n & 0_{n,m-n} \end{bmatrix} A^{-1},$$

and therefore L is surjective.

If $m \leq n$ and $L : \mathbb{V}^m \to \mathbb{W}^n$ has maximum rank, then the rank is m,

$$L = B \begin{bmatrix} I_m \\ 0_{n-m,m} \end{bmatrix} A^{-1},$$

and therefore L is injective.

It follows that if $m = n$ and $L : \mathbb{V}^m \to \mathbb{W}^n$ has maximum rank, then

$$L = B \begin{bmatrix} I_m \end{bmatrix} A^{-1}$$

is bijective.

E.2 Characterization of a linear subspace

Lemma E.2. *Given $0 \leq m \leq n$, an n-dimensional vector space $\mathbb{W}^n$, and $S \subset \mathbb{R}^n$, the following are equivalent:*

1. *S is an m-dimensional linear subspace of $\mathbb{W}^n$.*

2. *There exists a linear map* $L:\mathbb{R}^m \to \mathbb{W}^n$ *with rank* m *such that*

$$\operatorname{im} L = S.$$

3. *There exists a linear map* $M:\mathbb{W}^n \to \mathbb{R}^{n-m}$ *with rank* $n-m$ *such that*

$$\ker M = S.$$

The most basic example is

$$S = \{(x^1,\dots,x^m,0,\dots,0) \;:\; (x^1,\dots,x^m) \in \mathbb{R}^m\} \subset W = \mathbb{R}^n,$$

with the linear maps

$$\begin{aligned} L_{n,m} &: \mathbb{R}^m \to \mathbb{R}^n \\ (x^1,\dots,x^m) &\mapsto (x^1,\dots,x^m,0,\dots,0) \end{aligned}$$

and

$$\begin{aligned} M_{n,m} &: \mathbb{R}^n \to \mathbb{R}^{n-m} \\ (x^1,\dots,x^n) &\mapsto (x^{n-m+1},\dots,x^n). \end{aligned}$$

Lemma E.3. *If* $S \subset \mathbb{W}^n$ *is an* m*-dimensional subspace, then there exists a basis* E *of* $\mathbb{W}^n$ *such that the linear maps* $L = I_E L_{n,m}$ *and* $M = M_{n,m} I_E^{-1}$ *satisfy the properties in Lemma E.2.*

E.3 Linear subspace as a graph

A 1-dimensional linear subspace of $\mathbb{R}^2$ is given by an equation of the form

$$ax + by = 0.$$

When can this be written as a graph of a function, $y = f(x)$? The line cannot be vertical. Equivalently, b must be nonzero, and therefore we can solve for y,

$$y = -\frac{a}{b}x.$$

A 2-dimensional linear subspace of $\mathbb{R}^3$ is given by an equation of the form

$$ax + by + cz = 0.$$

Similarly, this is the graph of a function, if $c \neq 0$ and therefore we can solve for zm

$$z = -\frac{a}{c}x - \frac{b}{c}y.$$

A 2-dimensional linear subspace of $\mathbb{R}^4$ is given by two linear equations of the form

$$a_1^1 x^1 + a_2^1 x^2 + a_3^1 x^3 + a_4^1 x^4 = 0$$
$$a_1^2 x^1 + a_2^2 x^2 + a_3^2 x^3 + a_4^2 x^4 = 0$$

When can we solve for (x^3, x^4) in terms of (x^1, x^2)? First, note that the system can be rewritten in the form

$$\begin{bmatrix} A' & A'' \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = 0, \tag{E.1}$$

where

$$A' = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}$$
$$A'' = \begin{bmatrix} a_1^3 & a_2^3 \\ a_1^4 & a_2^4 \end{bmatrix}$$
$$x' = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$
$$x'' = \begin{bmatrix} x^3 \\ x^4 \end{bmatrix}.$$

We can therefore solve for $x' = (x^3, x^4)$ if and only if the matrix A'' is invertible. If so, we can multiply (E.1) on the left by $(A'')^{-1}$ to get an equivalent system of the form

$$\begin{bmatrix} (A'')^{-1}A' & I \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = 0,$$

which expands to

$$b_1^1 x^1 + b_2^1 x^2 + x^3 = 0$$
$$b_1^2 x^1 + b_2^2 x^2 + x^4 = 0,$$

where

$$\begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix} = (A'')^{-1}.$$

Recall that $\text{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ is the space of n-by-m matrices. In general, an m-dimensional subspace of $\mathbb{R}^n$, where $m < n$, is given by a linear system

$$Lx = \begin{bmatrix} L' & L'' \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix}, = 0, \tag{E.2}$$

where $x' \in \mathbb{R}^m$, $x'' \in \mathbb{R}^{n-m}$, $L' \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^{n-m})$, and $L'' \in \mathrm{Hom}(\mathbb{R}^{n-m}, \mathbb{R}^{n-m})$. This system can be rewritten equivalently as

$$x'' = Ax',$$

if and only if L'' is invertible. Multiplying (E.2) on the left by $(L'')^{-1}$, we get an equivalent system

$$\begin{bmatrix} (L'')^{-1}L' & I \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = 0,$$

and therefore, we can solve for x'' in terms of x':

$$x'' = -(L'')^{-1}L'x'.$$

E.4 Topology of the space of linear maps

Recall that, if we choose a basis for a vector space $\mathbb{V}^m$ and one for another vector space $\mathbb{W}^n$, then there is a linear isomorphism

$$\mathrm{Hom}(\mathbb{V}^m, \mathbb{W}^n) \simeq \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n).$$

There is a natural inner product defined on $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ given by

$$\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} A_i^j B_i^j,$$

which defines a norm, which in turn defines a distance function. We can therefore define open sets, convergence and limits of sequences, and continuity of functions and maps using the distance function. These definitions can then be transferred to the more abstract space $\mathrm{Hom}(\mathbb{V}^m, \mathbb{W}^n)$.

Let $\mathrm{GL}(\mathbb{V}^m) \subset \mathrm{Hom}(\mathbb{V}^m, \mathbb{V}^m)$ denote the space of invertible linear maps. For convenience we denote $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m)$ by $\mathrm{gl}(m)$ and $\mathrm{GL}(\mathbb{R}^m)$ by $\mathrm{GL}(m)$.

Recall that there is a norm defined on $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m)$.

Theorem E.4. *The following hold:*

1. $\mathrm{GL}(Vm)$ *is an open subset of* $\mathrm{Hom}(\mathbb{V}^m, \mathbb{V}^m)$.
2. *The map* $F : \mathrm{GL}(m) \to \mathrm{GL}(m)$, *where* $F(L) \to L^{-1}$ *is* C^∞.
3. *The differential of* F *is given by*

$$dF(L)\dot{L} = -L^{-1}\dot{L}L^{-1}.$$

Proof. The determinant of a matrix is a continuous function of the matrix. It suffices to prove this for $\mathbb{V}^m = \mathbb{R}^m$. First, we show that there exists $\delta > 0$ such that, for any $C \in \mathrm{gl}(m)$,

$$|C| < \delta \implies I + C \in \mathrm{GL}(m).$$

From this it follows that, given $A \in \mathrm{GL}(m)$, there exists $\delta_A > 0$ such thatm for any $B \in \mathrm{gl}(m)$,

$$|B| < \delta_A \implies A + b \in \mathrm{GL}(m).$$

This shows that $\mathrm{GL}(m)$ is an open subset of $\mathrm{gl}(m)$.

Recall that the inverse of $A \in \mathrm{GL}(m)$ is given by

$$A = \frac{1}{\det A} A^c,$$

where A^c is the cofactor matrix of A. Since both $\det A$ and the components of A^c are polynomial function of the components of A and $\det A \neq 0$ for every $A \in \mathrm{GL}(m)$, it follows that

□

E.5 Local versus global behavior of a 1-dimensional nonlinear map

Compare the following maps $F : \mathbb{R}^1 \to \mathbb{R}^1$:

$$F_0(x) = x$$
$$F_1(x) = x^3$$
$$F_2(x) = x^2$$
$$F_3(x) = \sin x$$

The maps F_0 and F_1 have inverse maps

$$F_0^{-1}(y) = y$$
$$F_1^{-1}(y) = y^{1/3},$$

but F_1 is not C^1 at $x = 0$. The map F_2 and F_3 have inverse maps, only if their domains are suitably restricted: If $\widehat{F}_2$ is F_2 restricted to $[0,\infty)$, then it has the inverse map $\widehat{F}_2^{-1} : [0,\infty \to [0,\infty)$, given by

$$\widehat{F}_2^{-1}(y) = \sqrt{y},$$

which is C^1 if restricted to $(0,\infty)$. If $\widehat{F}_3$ is F_3 restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then it has the inverse map $\widehat{F}_3 = \arcsin$, which is C^1 if restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

We shall focus on determining whether a map is locally invertible and the inverse is C^1. In other words, given x_0 in the domain of a map F, is there an open set O containing x_0 such that F restricted to O has a C^1 inverse map.

E.6 Normal form of C^1 maps with maximal rank

Definition E.5. If $O \subset \mathbb{V}^m$ and $F : O \to \mathbb{W}^n$ is a C^1 map, then F has *maximal rank* at $p \in O$, if its differential at $p \in O$, $dF(p) : \mathbb{V}^m \to \mathbb{W}^n$ has maximal rank.

A C^1 map $F : O \subset \mathbb{V}^m$ has *maximal rank* on O, if it has maximal rank at every $p \in O$.

Definition E.6. If $m \geq n$ and F has maximal rank, then F is a *submersion* of O.

If $m \leq n$ and F has maximal rank, then F is an *immersion* of O.

If $m = n$ and F has maximal rank, then F is a *local diffeomorphism* of O.

Definition E.7. Let $O \subset \mathbb{R}^m$ be open. A map $F : O \to \mathbb{R}^m$ is a C^1 *diffeomorphism*, if $F(O)$ is open, F is C^1, and there exists a C^1 map $G : F(O) \to O$ such that

$$\begin{aligned} G(F(x)) &= x, \ \forall\ x \in O \\ F(G(y)) &= y, \ \forall\ y \in F(O). \end{aligned}$$

In particular, if the map G exists, it is unique and the maps F and G are bijective.

If $F : O \to Rm$ is a C^1 map, it is a *local diffeomorphism* at $x_0 \in O$, if there exists an open neighborhood $O' \subset O$ of x_0 such that F restricted to O' is a C^1 diffeomorphism.

By the chain rule,

Lemma E.8. *If $F : O \to \mathbb{R}^m$ is a C^1 diffeomorphism at $x_0 \in O$, then the differential $dF(x) : \mathbb{R}^m \to \mathbb{R}^m$ is invertible for any $x \in O$.*

E.7 Contraction map lemma

We state this for an arbitrary vector space with a norm with respect to which the space is a complete topological space. In other words, any Cauchy sequence has a limit.

Lemma E.9. *Let $\mathbb{V}$ be a complete normed vector space and $O \subset \mathbb{V}$ an open subset. If $F : O \to O$ is a continuous map such that there exists $0 \leq c < 1$ such that for any $x_0, x_1 \in O$,*

$$|F(x_1) - F(x_0)| \leq c|x_1 - x_0|,$$

then there exists a unique $x \in O$ such that $F(x) = x$.

Proof. Given $x_0 \in O$, define a sequence as follows: For each $k \geq 0$, let

$$x_{k+1} = F(x_k).$$

It follows that, for each $k \geq 1$,

$$|x_{k+1} - x_k| = |F(x_k) - F(x_{k-1})| \leq c|x_k - x_{k-1}|.$$

From this, it follows by induction that, for any $0 \leq k \leq l$,

$$|x_{l+1} - x_l| \leq c^{l-k}|x_{k+1} - x_k|.$$

Therefore,

$$\begin{aligned} |x_l - x_k| &\leq |x_l - x_{l-1}| + \cdots + |x_{k+1} - x_k| \\ (c^{l-k} + \cdots + 1)|x_{k+1} - x_k| &\leq \frac{1 - c^{l-k+1}}{1 - c}|x_{k+1} - x_k| \\ &\leq c^k|x_1 - x_0|. \end{aligned}$$

For $\epsilon < 0$, let $N > 0$ satisfy

$$c^N |x_1 - x_0| = \epsilon.$$

Then for any $N \le k \le l$,

$$|x_l - x_k| \le c^k |x_1 - x_0| \le c^N |x_1 - x_0| \le \epsilon.$$

It follows that $(x_1, \cdots)$ is a Cauchy sequence and has a limit $x \in O$. By the continuity of F

$$x = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} F(x_k) = F(x).$$

Finally, if $F(x_1) = x_1$ and $F(x_2) = x_2$, then

$$|x_2 - x_1| = |F(x_2) - F(x_1)| \le c|x_2 - x_1|$$

which implies that

$$(1-c)|x_2 - x_1| \le 0,$$

and therefore $x_2 = x_1$. □

E.8 Inverse function theorem

The following is a version of the mean value theorem.

Lemma E.10. *If $F : O \to \mathbb{R}^n$ is a C^1 map and $x_0, x_1 \in O$ are such that the line segment joining them also lies in O, then*

$$F(x_1) - F(x_0) = \int_0^1 dF'((1-t)x_0 + tx_1)(x_1 - x_0)\,dt.$$

Proof.

$$F(x_1) - F(x_0) = \int_0^1 \frac{d}{dt} F((1-t)x_0 + t_1)\,dt$$
$$= \int_0^1 dF'((1-t)x_0 + tx_1)(x_1 - x_0)\,dt.$$

□

The following is a converse to Lemma E.8:

Lemma E.11. *Given $A \in \mathrm{GL}(m)$ and $v \in \mathbb{R}^m$,*

$$|Av| \le |A||v|.$$

Theorem E.12. *If $F : O \to \mathbb{R}^m$ is a C^1 map and $x_0 \in O$, then if $dF(x_0) : \mathbb{R}^m \to \mathbb{R}^m$ is an invertible linear map, then there exists an open neighborhood $O' \subset O$ of x_0 such that F restricted to O' is a C^1 diffeomorphism. In particular, there $F(O')$ is open and there exists a C^1 map $G : F(O') \to O$ such that*

$$F(G(y)) = y,\ \forall\ y \in F(O') \text{ and } G(F(x)) = x,\ \forall\ x \in O'.$$

Proof. Let $L:\mathbb{R}^m \to \mathbb{R}^m$ be the inverse linear map to $dF(x_0)$ and

$$O' = B(x_0, \delta),$$

where

$$\delta = \frac{1}{2}|L|^{-1}.$$

Given $y \in F(O')$, if

$$\Phi(x) = x - L(y - F(x)),$$

then $y = F(x)$ if and only if $\Phi(x) = x$, so it suffices to show that $\Phi : O' \to O'$ is a contraction mapping. First, note that, for any $x_1, x_2 \in O'$,

$$\Phi(x_2) - \Phi(x_1) = \int_0^1 dF((1-t)x_2 + tx_1)(x_2 - x_1)\,dt$$
$$x_2 - x_1 = LdF(x_0)(x_2 - x_1)$$

and therefore

$$\begin{aligned}\Phi(x_2) - \Phi(x_1) &= LdF(x_0)(x_2 - x_1) + L^{-1}(F(x_2) - F(x_1)) \\ &= L\int_0^1 (dF(x_0) - dF((1-t)x_2 + tx_1))(x_2 - x_1)\,dt\end{aligned}$$

It follows that

$$\begin{aligned}|\Phi(x_2) - \Phi(x_1)| &\le |L|\frac{1}{2}|L|^{-1}|x_2 - x_1| \\ &\le \frac{1}{2}|x_2 - x_1|.\end{aligned}$$

□

E.9 Normal forms of nonlinear maps

Definition E.13. Given an open set $O \subset \mathbb{R}^m$ and $m \le n$, a C^1 map $F : O \to \mathbb{R}^n$ is an *immersion*, if for any $x \in O$, the differential $dF(x) : \mathbb{R}^m \to \mathbb{R}^n$ is injective. This is equivalent to saying that $m \le n$ and the rank of $dF(x)$ is m.

If F is also injective, then it is an *embedding*.

Theorem E.14. *If $F : O \to \mathbb{R}^n$ is a C^1 immersion, then for each $x \in O$, there exists an open neighborhood $O' \subset O$ of x and diffeomorphisms $A : A^{-1}(O') \to O'$ and $B : B^{-1}(F(O')) \to F(O')$ such that*

$$F = B \circ L_{n,m} \circ A^{-1}$$

Definition E.15. Given an open set $O \subset \mathbb{R}^n$, a C^1 map $F : O \to \mathbb{R}^{n-m}$ is a *submersion*, if for any $x \in O$, the differential $dF(x) : \mathbb{R}^n \to \mathbb{R}^{n-m}$ has rank $n - m$. This is equivalent to $dF(x)$ being surjective.

Theorem E.16. *If $F : O \to \mathbb{R}^{n-m}$ is a C^1 submersion, then for each $x \in O$, there exists an open neighborhood $O' \subset O$ of x and diffeomorphisms $A : A^{-1}(O') \to O'$ and $B : B^{-1}(F(O')) \to F(O')$ such that*

$$F = B \circ M_{n,m} \circ A^{-1}$$

E.10 Partition of unity

Let M be an m-dimensional C^2 manifold. Therefore, there exists a countable locally finite covering of M by coordinate charts $x_i : O_i \to B(0,1) \subset \mathbb{R}^m$. In particular, each x_i is bijective and, for any i and j, $x_j \circ x_i^{-1} : x_i(O_i \cap O_j) \to x_j(O_i \cap O_j)$ is a C^2 diffeomorphism.

Let $\phi : B(0,1) \to [0,\infty)$ be given by

$$\phi(x) = \frac{1+\cos(\pi|x|)}{2}.$$

Let $\phi_i : M \to [0,\infty)$ be the C^2 function given by

$$\psi_i(p) = \begin{cases} \phi(x_i(p)) & \text{, if } p \in O_i \\ 0 & = \text{, otherwise} \end{cases}$$

Observe that

$$\psi = \sum_i \psi_i$$

is strictly positive and therefore, the functions

$$\chi_i = \frac{\psi_i}{\psi}$$

are well-defined C^2 functions on M such that

$$\sum_i \chi_i = 1.$$

Bibliography