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Chapter 1
Introduction

The goal of these notes is to provide an introduction to differential geometry, first by studying geometric properties of curves and surfaces in Euclidean 3-space. Guided by what we learn there, we develop the modern abstract theory of differential geometry.

The approach taken here is radically different from previous approaches. Instead of working initially with $\mathbb{R}^n$ exclusively and introducing the modern abstract perspective later, I have chosen to introduce and use abstraction from the start. The philosophy here is to use abstraction whenever possible, and switch back to using coordinates on $\mathbb{R}^n$ only as needed.

Why do it this way, especially since abstraction can be difficult to understand? First, I believe that, although coordinates are at first sight easier to understand, they interfere with developing a clear conceptual understanding of the ideas. My view is that the abstract approach reflects more closely the geometric intuition driving the subject. Second, although coordinates are essential for proving some of the fundamental technical theorems, such as the implicit function theorem, most of the lemmas needed for laying the foundations of the subject are more easily proved using the abstract framework than using coordinates.

Last, the abstract perspective of mathematics is extremely useful, if not essential, for learning and doing advanced mathematics, even applied mathematics. Since it is so difficult to learn, it requires for most of us several attempts before we get it. The sooner you start struggling with it, the better.

Warning: All functions and maps are assumed to be continuous, $C^1$, or $C^2$ (we'll discuss what this means later), depending on the context. Many of the results presented do not hold if the functions or maps are assumed to be only differentiable.
Chapter 2

Flat geometry

Flat plane geometry was developed by Euclid using an axiomatic approach. He formulated a set of definitions and axioms (assumptions) and derived geometric theorems from them using only deductive logic. Later, Descartes observed that Euclidean plane can be represented using coordinates, and the theorems of Euclidean geometry proved using basic algebra. Some time after that, it was recognized that Descartes’ approach can be done using abstract algebra.

But what is geometry? What is a geometric space, and when is a statement about the space called geometric? An initial answer comes from physics. When we study physics, we have to make measurements, which in turn require using units. For example, for length we can use inches, centimeters, or other units. However, we believe that the laws of physics should not depend on the units used. This means that a law of physics should hold, no matter what units are used. Ideally, one should be able to state the laws of physics without using any units at all. This in fact is always possible by expressing them using unitless ratios.

In particular, in Newtonian physics it is assumed that empty space satisfies the properties of Euclidean 3-space. Therefore, any law of physics has to be logically consistent with the axioms and theorems of Euclidean geometry. On the other hand, spatial measurements require units, which in higher dimensions, includes a way to measure directions. The most convenient way to do this is to use Cartesian coordinates. However, the statements of physical laws and properties should not depend on where the origin is placed and what directions the coordinate axes point in.

Euclidean geometry can therefore be defined in one of two equivalent ways. It is the study of rigorous logical consequences of the Euclidean axioms. Or it is the study of theorems about Euclidean space, where the theorems and proofs might be stated using Cartesian coordinates but remain valid if the coordinates are changed by either shifting the origin to a different point in space or rotating the coordinate axes.

After the introduction of Cartesian coordinates, Euclid’s original axiomatic approach has been rarely used, because the proofs are often quite subtle and difficult to find. On the other hand, although proofs using Cartesian coordinates are usually more straightforward, one must also verify that the theorem proved remains valid under a change of coordinates. The modern abstract algebraic approach allows theorems to be stated without any reference to coordinates and sometimes short straightforward proofs.

In this chapter we briefly review the three different approaches. More details can be
found in the book of Stillwell [1].

2.1 Axiomatic geometry

There are several different formulations of the axioms for Euclidean plane geometry. In all of them one starts with points, lines, and circles. Euclid himself first defined what are known as straightedge and compass constructions and then additional axioms. Others who formulated Euclidean geometry in terms of independent axioms include Hilbert, Birkhoff, and Tarski. Presented here are Hilbert’s axioms.

2.1.1 Axioms of the line

We begin with axioms that specify the properties of a line and points on the line. Note that these axioms never refer to an origin, the length of a line segment, or numbers at all.

Order.

Concept of order. Given any three distinct points lying on a line, exactly one lies between the other two.

Existence of betweenness. Given two distinct points on a line, there exists a point on the line between them.

Linear congruence.

Definition of oriented line segment. Given any two points A and B, the oriented line segment \( \overline{AB} \) is defined to be the set containing A, B, and all points between A and B, along with the designation that A is the start and B is the end of \( \overline{AB} \). In particular, if A \( \neq \) B, then \( \overline{AB} \neq \overline{BA} \).

- Definition of ray. Given two distinct points A and B, the ray \( \overrightarrow{AB} \) is defined to be the union of \( \overline{AB} \) and the set of all points for which B lies between A and the point in question.

- Directions of rays. Two rays point in the same direction, if one contains the other.

- Directions of oriented line segments. Two oriented line segments \( \overline{AB} \) and \( \overline{CD} \) point in the same direction, if the rays \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) point in the same direction.

Congruence of oriented line segments. There is an equivalence relation called congruence on the set of oriented line segments and denoted by \( \cong \).

Existence of congruent line segments. Given a line segment \( \overline{AB} \) and a point C, there exists a unique point D such that \( \overline{AB} \cong \overline{CD} \).

Concatenation of line segments. Given line segments \( \overline{AB} \) and \( \overline{CD} \), let \( \overline{AB} \lor \overline{CD} \) denote the line segment \( \overline{AE} \), where \( \overline{BE} \cong \overline{CD} \).
Multiple of a line segment. Given a line segment $\overline{AB}$, let $1\overline{AB} = \overline{AB}$. If $n \in \mathbb{Z}^+$, let 

$$(n + 1)\overline{AB} = n\overline{AB} \lor \overline{AB}.$$ 

Additivity of congruence. Given points $A, B, C, D, E, F, G, H$, 

$$\overline{AB} \cong \overline{EF} \text{ and } \overline{CD} \cong \overline{GH} \implies \overline{AB} \lor \overline{CD} \cong \overline{EF} \lor \overline{GH}.$$ 

Topology of a line

Relative length. A line segment $\overline{AB}$ is longer than a line segment $\overline{CD}$, if $D$ lies between $C$ and $E$, where $\overline{CE}$ is the line segment in the ray $\overline{CD}$ that is congruent to $\overline{AB}$.

Archimedean axiom. For any line segments $\overline{AB}$ and $\overline{CD}$, there exists $n \in \mathbb{Z}^+$ such that $n\overline{AB}$ is longer than $\overline{CD}$.

Completeness. Let $S, T$ be nonempty subsets of a line $\ell$, such that $S \cup T = \ell$ and no point in one subset lies between two points in the other. Then there exists a unique point $P$ such that for any points $A \in S \backslash \{P\}$ and $B \in T \backslash \{P\}$, $P$ lies between $A$ and $B$.

2.1.2 Axiomatic plane geometry

Incidence.

Unique line containing two points. For any two distinct points $A$ and $B$, there exists a unique line $\overrightarrow{AB}$ passing through them.

Line has at least two points. Every line contains at least two distinct points.

Definition of collinearity. A set of points is collinear, if it is contained in a line.

Dimension is greater than 1. There exist three points that are not collinear.

Parallel axiom. For each line $\ell$ and point $P$ not on $\ell$, there exists a unique line through $P$ that does not intersect $\ell$.

Pasch’s axiom. Suppose the points $A, B, C$ are not collinear and $\ell$ is a line that does not pass through any of these points. If $\ell$ contains a point between $A$ and $B$, then it also contains either a point between $A$ and $C$ or a point between $B$ and $C$, but not both.

Angles.

Concept of angle. Any two distinct rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ defines an angle $\angle ABC$.

Concept of congruence. There is an equivalence relation called congruence on the set of angles and denoted $\cong$.

Existence of congruent angles. Given any angle $\angle ABC$ and a ray $\overrightarrow{ED}$, there exists a unique ray $\overrightarrow{EF}$ such that $\angle DEF \cong \angle ABC$. 
Definition of triangle. A triangle consists of non-collinear points $A$, $B$, $C$ and is denoted by $\triangle ABC$.

Congruence of triangles. Two triangles $\triangle ABC$ and $\triangle DEF$ are congruent, $\triangle ABC \cong \triangle DEF$, if
\[
\overline{AB} \cong \overline{DE}, \quad \overline{BC} \cong \overline{EF}, \quad \overline{CA} \cong \overline{FD},
\]
and
\[
\angle ABC \cong \angle DEF, \quad \angle BCA \cong \angle EFD, \quad \angle CAB \cong \angle FDE.
\]

SAS. If the triangles $\triangle ABC$ and $\triangle DEF$ satisfy
\[
\overline{AB} \cong \overline{DE}, \quad \overline{BC} \cong \overline{EF}, \quad \text{and} \quad \angle ABC \cong \angle DEF,
\]
then they are congruent.

Circles

Definition of a circle. A circle centered at $A$ is a set of points such that for any two points $B$ and $C$ lying in the circle, $\overline{AB} \cong \overline{AC}$.

Interior of a circle. A point $C$ lies inside a circle centered at $A$, if there exists a point $B$ in the circle such that $C$ lies between $A$ and $B$. A point lies outside a circle if it does not lie on or inside the circle.

Intersection of circles. If one circle contains both points inside and outside another circle, then the two circles intersect.

2.1.3 Consequences

Length and angles
Areas of triangles and parallelograms
Pythagorean theorem
Arithmetic from geometry

2.2 Flat geometry using coordinates

2.2.1 Cartesian line

Given a line $\ell$, choose a point $O \in \ell$, call it the origin, and label it by 0. Choose another point $E \neq O$ on the line and label it by 1. There exists a unique point $F \neq O$ such that $\overline{OE} \cong \overline{EF}$. Label $F$ by 2. Continuing by induction, each nonnegative integer labels a unique point.

There exists a unique point $E' \neq E$ such that $\overline{EO} \cong \overline{OE}$. Label it by $-1$. Continuing by induction, each integer labels a unique point.
Each unit segment can be bisected. Each resulting smaller segment can also be bisected. Continuing by induction, each fraction whose denominator is a power of 2 labels a unique point.

By the completeness axiom, every real number labels a unique point, and every point is labeled by a unique real number. We call this the real line and denote it by \( \mathbb{R} \).

Given any two points \( a, b \in \mathbb{R} \), the line segment \( \overline{ab} \) is congruent to a line segment \( \overline{nc} \), where \( c \geq 0 \), if and only if \( c = |b - a| \). We can therefore define the length of a line segment \( \overline{ab} \) to be \( |b - a| \) and the distance between \( a \) and \( b \) to be \( |b - a| \).

### 2.2.2 Cartesian plane

The Cartesian plane can be defined from the Euclidean plane as follows. Start with a point \( O \) on the plane. Choose a line passing through \( O \) and call it the \( x \)-axis. Choose a second line orthogonal to the \( x \)-axis and call it the \( y \)-axis. Choose a point \( E_1 = 0 \) on the \( x \)-axis and a point \( E_2 = 0 \) on the \( y \)-axis such that \( OE_1 = OE_2 \). By the construction of the Cartesian line, each axis is a real line, where \( E_1 \) and \( E_2 \) are labeled by 1.

For each point \( P \) in the plane, it follows by the parallel axiom that there exists a unique line passing through \( P \) and parallel to the \( y \)-axis. That line must intersect the \( x \)-axis. Let \( a \) denote the real number labeling the point of intersection. Similarly, there is a unique line through \( P \) that is parallel to the \( x \)-axis. Let \( b \) label the intersection of that line with the \( y \)-axis. Label \( P \) by the ordered pair \((a, b)\). Let \( \mathbb{R}^2 \) denote the set of all possible ordered pairs \((a, b)\), where \( a, b \in \mathbb{R} \).

Conversely, given an ordered pair \((a, b)\), there exist a unique line parallel to the \( y \)-axis and passing through \( a \) on the \( x \)-axis and a second unique line parallel to the \( x \)-axis and passing through \( b \) on the \( y \)-axis. These two lines must intersect, and the point of intersection is the point labeled by \((a, b)\).

### 2.2.3 Length and distance

Given points \( P_1 = (a_1, b_1) \) and \( P_2 = (a_2, b_2) \), there exists a unique \( c \geq 0 \) such that \( \overline{P_1P_2} \cong \overline{0c} \). We define the length of \( \overline{P_1P_2} \) and the distance between \( P_1 \) and \( P_2 \) to be \( c \). If we let \( Q = (a_1, b_2) \), then \( \triangle P_1QP_2 \) is a right triangle. By the Pythagorean theorem,

\[
\ell(\overline{P_1P_2})^2 = (\ell(\overline{P_1Q}))^2 + (\ell(\overline{QP_2}))^2.
\]

This is equivalent to

\[
(d(P_1, P_2))^2 = (a_2 - a_1)^2 + (b_2 - b_1)^2.
\]

### 2.2.4 Angle via dot product

Recall that, if \( P_0 = (a_0, b_0) \), \( P_1 = (a_1, b_1) \), and \( P_2 = (a_2, b_2) \), then

\[
(P_1 - P_0) \cdot (P_2 - P_0) = (a_1 - a_0)(a_2 - a_0) + (b_1 - b_0)(b_2 - b_0).
\]

By the law of cosines,

\[
(P_1 - P_0) \cdot (P_2 - P_0) = d_1 d_2 \cos \theta,
\]
, where
\[
\begin{align*}
    d_1 &= \sqrt{(a_1 - a_0)^2 + (b_1 - b_0)^2} \\
    d_2 &= \sqrt{(a_2 - a_0)^2 + (b_2 - b_0)^2} \\
    \theta &= \angle P_1 P_0 P_2.
\end{align*}
\]

### 2.3 Abstract vector space with an inner product

Let \( \mathbb{V}^m \) be an abstract \( m \)-dimensional vector space. A function
\[
V \times V \rightarrow \mathbb{R} \\
(v_1, v_2) \mapsto \langle v_1, v_2 \rangle
\]
is called an *inner product*, if the following hold for any \( v, v_1, v_2 \in V \) and \( c \in \mathbb{R} \):
\[
\begin{align*}
    \langle v_2, v_1 \rangle &= \langle v_1, v_2 \rangle \quad \text{(symmetry)} \\
    \langle v, v_1 + v_2 \rangle &= \langle v, v_1 \rangle + \langle v, v_2 \rangle \\
    \langle v_1, cv_2 \rangle &= c \langle v_1, v_2 \rangle \\
    \langle v, v \rangle &\geq 0 \\
    \langle v, v \rangle &= 0 \iff v = 0
\end{align*}
\]
(positive definiteness).

We can then define the length of a vector to be
\[
|v| = \sqrt{\langle v, v \rangle}
\]
and the angle \( \theta \) between two nonzero vectors \( v_1 \) and \( v_2 \) by
\[
\cos \theta = \frac{\langle v_1, v_2 \rangle}{|v_1||v_2|}
\]
The basic example of an abstract \( m \)-dimensional vector space is \( \mathbb{R}^m \), and the basic example of an inner product is the dot product,
\[
(v_1^1, \ldots, v_1^m) \cdot (v_2^1, \ldots, v_2^m) = v_1^1 v_2^1 + \cdots + v_1^m v_2^m
\]
We will see that this is essentially the only possible example. Nevertheless, it is often useful to forget about the Cartesian coordinates and use only the abstract definitions.

It is now straightforward to show that if you start with the Euclidean axioms, choose a point and call it the origin, then you can define vector addition and scalar multiplication on the Euclidean space, where the Euclidean axioms imply the properties listed above and the definitions of length and distance above are consistent with the definitions implied by the axioms.

Conversely, it is also straightforward to show that the properties of a vector space with an inner product imply the Euclidean axioms.
2.4 Euclidean geometry on an affine space

The vector space approach to Euclidean geometry requires choosing a special point in space and designating it as the origin. This can be avoided using affine geometry. In this approach, there is a space of points (which are not the same as vectors), called affine space, and an associated vector space, which we will call the tangent space, with an inner product. A detailed presentation is given in Chapter 4. Again, it can be shown that the Euclidean axioms are equivalent to the definition of an affine space with an inner product on its tangent space.
Chapter 3

Vector Spaces and Linear Maps

This is a review of concepts and definitions that you should have already learned in linear algebra.

3.1 Definition of a vector space

A vector space can be defined with respect to any field $\mathcal{F}$ such as the rationals $\mathbb{Q}$, reals $\mathbb{R}$, or complex numbers $\mathbb{C}$. We, however, will restrict our attention to vector spaces with respect to the reals.

**Definition 3.1.** A vector space over $\mathbb{R}$ that has the operations of addition,

$$v_1, v_2 \in V \mapsto v_1 + v_2 \in V,$$

and scalar multiplication

$$r \in \mathbb{R}, v \in V \mapsto rv \in V,$$

which satisfy the following properties:

\begin{align*}
(1) & \quad (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3), \quad \forall v_1, v_2, v_3 \in V \\
(2) & \quad v_1 + v_2 = v_2 + v_1, \quad \forall v_1, v_2 \in V \\
(3) & \quad \forall v \in V, \exists -v \in V \text{ such that } v + (-v) = 0 \\
(4) & \quad (r_1 r_2)v = r_1(r_2v), \quad \forall r_1, r_2 \in \mathbb{R}, \forall v \in V \\
(5) & \quad r(v_1 + v_2) = rv_1 + rv_2, \quad \forall r \in \mathbb{R}, \forall v_1, v_2 \in V \\
& \quad \exists \mathbf{0} \in V \text{ such that } \mathbf{0} = 0v, \quad \forall v \in V \\
& \quad 1v = v, \quad \forall v \in V.
\end{align*}

If we denote $-v = (-1)v$, then, by the properties above,

$$v + (-v) = 1v + (-1)v = (1 + (-1))v = 0v = \mathbf{0}.$$

For convenience we will denote $\mathbf{0}$ by simply 0. It will always be clear, from the context, whether 0 represents the scalar 0 or the vector $\mathbf{0}$. 

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3.2 Examples of vector spaces

The most basic example of a vector space is $\mathbb{R}^m$. However, depending on the context, $\mathbb{R}^m$ can play different roles, sometimes as a vector space, sometimes as an affine space (defined later), and sometimes as a dual vector space. It is useful to distinguish clearly between these different types of $\mathbb{R}^m$. We therefore shall use the following notation.

Let

$$\vec{R}^m = \{ \vec{v} = (v^1, \ldots, v^m), \text{ where } v^1, \ldots, v^m \in \mathbb{R} \}.$$  

We shall also sometimes write a vector $\vec{v}$ as a column vector,

$$\vec{v} = (v^1, \ldots, v^m) = \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix}.$$

Other examples of vector spaces are

1. The set of all solutions to a system of homogeneous linear equations.

2. Polynomials with coefficients in $\mathbb{R}$.

3. Continuous functions $f : \mathbb{R} \to \mathbb{R}$

3.3 Linearly independent vectors

Definition 3.2. A finite set of vectors, $\{v_1, \ldots, v_k\} \subset V$ is linearly independent, if, for any scalars $a^1, \ldots, a^k \in \mathbb{R}$,

$$a^1v_1 + \cdots + a^kv_k = 0 \implies a^1 = \cdots = a^k = 0.$$

Definition 3.3. Given a subset $S \subset V$, the span of $S$ in $V$ is defined to be

$$[S] = \{a^1v_1 + \cdots + a^kv_k : \forall a^1, \ldots, a^k \in \mathbb{R}, v_1, \ldots, v_k \in S, k > 0 \}.$$

Note that the set $S$ need not be finite.

Lemma 3.4. Any nonempty finite set $S \subset V$, where $S \neq \{0\}$, contains a linearly independent subset $T$ such that $[T] = [S]$.

Lemma 3.5. If $S = \{v_1, \ldots, v_k\} \subset V$ is linearly independent, then, for any $v \in [S]$, there exists a unique set of scalars $a^1, \ldots, a^k \in \mathbb{R}$ such that

$$v = a^1v_1 + \cdots + a^kv_k.$$
3.4 Basis and dimension of a vector space

Definition 3.6. A vector space $V$ is finite dimensional, if there exists a finite set $S = \{v_1, \ldots, v_N\} \subset V$ such that $[S] = V$.

Definition 3.7. An ordered list of vectors, $E = (e_1, \ldots, e_m)$, is a basis of $V$, if the vectors are linearly independent and span $V$, i.e., $[E] = V$.

Lemma 3.8. Any finite dimensional vector space has at least one basis.

Lemma 3.9. If $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_n)$ are both bases of $V$, then $m = n$.

Proof. Since $e_1, \ldots, e_m$ is a basis, there is a unique matrix $A$ such that, for each $1 \leq i \leq n$,

$$f_i = A^1_i e_1 + \cdots + A^m_i e_m.$$ 

On the other hand, since $f_1, \ldots, f_n$ is a basis, there exists a unique matrix $B$ such that, for each $1 \leq p \leq m$,

$$e_p = B^1_p f_1 + \cdots + B^m_p f_n.$$ 

Therefore, for each $1 \leq p \leq m$,

$$e_p = \sum_{i=1}^n B^i_p f_i = \sum_{i=1}^n B^i_p \left( \sum_{q=1}^m A^q_i e_q \right) = \sum_{q=1}^m \left( \sum_{i=1}^n B^i_p A^q_i \right) e_q.$$

It follows that

$$\sum_{i=1}^n B^i_p A^q_i = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases} \quad (3.8)$$

The same calculation above with the two bases switched gives

$$\sum_{p=1}^m A^p_i B^j_p = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \quad (3.9)$$

Therefore, first by (3.8) and then by (3.9),

$$m = \sum_{p=1}^m \sum_{i=1}^n B^i_p A^p_i = \sum_{i=1}^n \sum_{p=1}^m A^p_i B^i_p = n.$$

Definition 3.10. The dimension of a vector space $V$ is $m$, if it has a basis with $m$ elements. The vector space $V = \{0\}$ is said to be 0-dimensional.
Lemma 3.11. If \( E = (e_1, \ldots, e_m) \) is a basis of \( \mathbb{V}^m \), then, for each \( v \in \mathbb{V}^m \), there is a unique \( A = (a^1, \ldots, a^m) \in \mathbb{R}^m \) such that

\[
v = e_i a^i.
\]

It is convenient here to write everything using formal matrix notation. First, we view a basis as a row matrix of vectors (which is why the indices are subscripts):

\[
E = (e_1, \ldots, e_m) = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix},
\]

and an element \( A \in \mathbb{R}^m \) as a column matrix (which is why the indices are superscripts):

\[
A = \begin{bmatrix} a^1 \\
\vdots \\
. \\
a^m \end{bmatrix}.
\]

The vector \( v \) can now be written as

\[
v = e_i a^i = EA.
\]

This notation will simplify many of the formulas later.

### 3.5 Linear maps

A linear map from a vector space \( \mathbb{V} \) to a vector space \( \mathbb{W} \) (which could be \( \mathbb{V} \) itself) that preserves addition and scaling. In particular, it is a map denoted

\[
L : \mathbb{V} \to \mathbb{W}
\]

\[
v \mapsto L(v),
\]

where, for any \( v_1, v_2, v \in \mathbb{V} \) and \( a \in \mathbb{R} \),

\[
L(v_1 + v_2) = Lv_1 + Lv_2 \\
L(av) = a(Lv).
\]

**Example.** Let \( \mathbb{V} = \mathbb{R}^m, \mathbb{W} = \mathbb{R}^n \), and \( L \) is defined using an \( n \times m \) matrix \( A \),

\[
L : \mathbb{R}^m \to \mathbb{R}^n
\]

\[
\tilde{v} = \begin{bmatrix} v^1 \\
\vdots \\
v^m \end{bmatrix} \mapsto A\tilde{v} = \begin{bmatrix} A^1 v^1 \\
\vdots \\
A^m v^m \end{bmatrix} = \begin{bmatrix} A^1 v^1, \ldots, A^m v^m \end{bmatrix}.
\]

**Example.** Let \( \mathcal{P} \) be the set of polynomials with coefficients in \( \mathbb{R} \). Differentiation defines a linear map \( L : \mathcal{P} \to \mathcal{P} \), where, for each \( p \in \mathcal{P} \),

\[
L(p) = p'.
\]

**Definition 3.12.** A linear map \( L : \mathbb{V} \to \mathbb{W} \) is *injective* or 1-1, if, for any \( v_1, v_2 \in \mathbb{V} \),

\[
Lv_1 = Lv_2 \iff v_1 = v_2.
\]

It is *surjective* or *onto*, if, for any \( w \in \mathbb{W} \), there exists \( v \in \mathbb{V} \) such that \( Lv = w \). It is an *isomorphism* if it is both injective and surjective.
3.5. LINEAR MAPS

3.5.1 Isomorphism of an $m$-dimensional vector space with $\mathbb{R}^m$

**Lemma 3.13.** Given a basis $E = (e_1, \ldots, e_m)$ of $\mathbb{V}^m$, the map

$$I_E : \mathbb{R}^m \to \mathbb{V}^m$$

$$\langle v^1, \ldots, v^m \rangle \mapsto v^1 e_1 + \cdots + v^m e_m,$$

is a linear isomorphism.

3.5.2 The space of linear maps

Let $\text{Hom}(\mathbb{V}^m, \mathbb{W}^n)$ denote the space of all linear maps from $\mathbb{V}^m$ to $\mathbb{W}^n$. We can define the sum of two linear maps $L_1, L_2 \in \text{Hom}(\mathbb{V}^m, \mathbb{W}^n)$ to be $M : \mathbb{V}^m \to \mathbb{W}^n$, where

$$M(v) = L_1(v) + L_2(v), \quad \forall v \in \mathbb{V}^m.$$ 

It is straightforward to verify that $M$ is linear and therefore an element of $\text{Hom}(\mathbb{V}^m, \mathbb{W}^n)$. We shall denote it by $L_1 + L_2$. Similar, the scalar multiple of $L \in \text{Hom}(\mathbb{V}^m, \mathbb{W}^n)$ by $r \in \mathbb{R}$ is the map

$$(rL)(v) = r(L(v)), \quad \forall v \in \mathbb{V}^m.$$ 

It is straightforward to show that using these definitions, $\text{Hom}(\mathbb{V}^m, \mathbb{W}^n)$ is a vector space.

Let $\mathbb{M}(n, m) = \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ denote the set of $n$-by-$m$ matrices.

Given a basis $E = (e_1, \ldots, e_m)$ of $\mathbb{V}^m$ and a basis $F = (f_1, \ldots, f_n)$ of $\mathbb{W}^n$, then for any linear map $L : \mathbb{V}^m \to \mathbb{W}^n$, there is a sequence of maps

$$\mathbb{R}^m \xrightarrow{I_E} \mathbb{V}^m \xrightarrow{L} \mathbb{W}^n \xrightarrow{I_F^{-1}} \mathbb{R}^n.$$ 

This defines a map

$$I_{E,F} : \text{Hom}(\mathbb{V}^m, \mathbb{W}^n) \simeq \mathbb{M}(n, m)$$

$$L \mapsto I_F^{-1} \circ L \circ I_E$$

**Lemma 3.14.** $I_{E,F}$ is a linear isomorphism, and therefore $\text{Hom}(\mathbb{V}^m, \mathbb{W}^n)$ is an $mn$-dimensional vector space.

3.5.3 Subspaces and quotient spaces

**Definition 3.15.** A set $S \subseteq \mathbb{V}$ is a subspace of $\mathbb{V}$, if it is closed under addition and scalar multiplication. As a consequence, using the same addition and scalar multiplication operations as $\mathbb{V}$, the subspace $S$ is itself a vector space.

**Example.** The set

$$S = \{(x, y) : x + y = 0\} \subset \mathbb{R}^2$$

is a subspace.
Example. The set
\[ T = \{(x, y) : x + y = 1\} \subset \mathbb{R}^2 \]
is not a subspace. However, you can define different addition and scalar operations for which
\( S \) is a vector space, namely
\[
(x_1, y_1) + (x_2, y_2) = (1, 0) + [(x_1, y_1) - (1, 0)] + [(x_2, y_2) - (1, 0)]
= (x_1 + x_2 - 1, y_1 + y_2)
\]
\[
a \cdot (x, y) = (1, 0) + a[(x, y) - (1, 0)]
= (1 + a(x - 1), ay).
\]

If \( S \subset V \) is a linear subspace and \( v \in V \), let
\[ v + S = \{v + s : s \in S\}. \]
If the subspace \( S \) is clearly indicated, then we can also write \([v] = v + S\). Note that
\[ [v] = [w] \iff w - v \in S. \]

**Definition 3.16.** Let
\[ V/S = \{v + S : v \in V\}. \]

There is a natural short exact sequence (see below)
\[ 0 \to S \to V \to V/S \to 0 \quad (3.10) \]

**Definition 3.17.** A chain of two maps
\[ A \xrightarrow{f} B \xrightarrow{g} C \]
is exact, if \( \text{im } f = \ker g \). A longer chain of maps is exact, if any two consecutive maps are exact.

The exactness of the sequence \((3.10)\) means that the inclusion map \( S \subset V \) is injective, the kernel of the map \( V \to V/S \) is \( S \), and the map \( V \to V/S \) is surjective.

**Lemma 3.18.** Given a subspace \( S \subset V \),
\[ \dim V = \dim S + \dim V/S. \]

### 3.5.4 Kernel and image of a linear map

**Definition 3.19.** Given a linear map \( L : V \to W \), the kernel, image, and cokernel of \( L \) are defined to be
\[
\ker L = \{v \in V : L(v) = 0\} \subset V
\]
\[
\text{im } L = \{L(v) : v \in V\} \subset W
\]
\[
\text{coker } L = W/\text{im } L.
\]
Lemma 3.20.

\[
\begin{align*}
\ker L & \text{ is a subspace of } \mathbb{V} \\
\operatorname{im} L & \text{ is a subspace of } \mathbb{W} \\
\dim \ker L + \dim \operatorname{im} L & = \dim \mathbb{V} \\
\dim \operatorname{im} L + \dim \operatorname{coker} L & = \dim \mathbb{W}.
\end{align*}
\]

Definition 3.21. The rank of \( L \) is defined to be

\[ \operatorname{rank} L = \dim \operatorname{im} L. \]

Therefore, the identities above can also be written as

\[
\begin{align*}
\dim \ker L + \operatorname{rank} L & = \dim \mathbb{V} \\
\operatorname{rank} L + \dim \operatorname{coker} L & = \dim \mathbb{W}.
\end{align*}
\]

3.6 Normal forms of a linear transformation

Lemma 3.22. Let \( L : \mathbb{V} \to \mathbb{W} \) be a linear map, and \( F = (f_1, \ldots, f_n) \) a basis of \( \mathbb{W}^n \). There exists a basis \( E = (e_1, \ldots, e_m) \) and a reordering of the basis \( F \) such that the matrix \( M = I_{E,F}(L) \) satisfies the following:

\[
M^i_j = \begin{cases} 
\delta^i_j & \text{if } 1 \leq i, j \leq k \\
0 & \text{if } k + 1 \leq i \leq n \text{ and } 1 \leq j \leq m, 
\end{cases} \tag{3.11}
\]

where \( k = \operatorname{rank} L \).

Proof. There exists a reordering of \( F \) such that \( ([f_{k+1}], \ldots, [f_n]) \) is a basis of \( \operatorname{coker} L \). For each \( 1 \leq i \leq k \), there exists \( A^i_\mu \), \( k + 1 \leq \mu \leq n \), such that the vectors \( f_i + A^i_\mu f_\mu \) are a basis of \( \operatorname{im} L \). Therefore, there exist \( e_1, \ldots, e_k \) such that \( L(e_i) = f_i + A^i_\mu f_\mu \). Let \( (e_{k+1}, \ldots, e_m) \) be a basis of \( \ker L \). Note that \( (e_1, \ldots, e_m) \) is a basis of \( \mathbb{V} \). The matrix \( M = I_{E,F}(L) \) now satisfies (3.11) \( \square \)

Corollary 3.23. For any \( n \)-by-\( m \) matrix \( M \), there exists an invertible \( m \)-by-\( m \) matrix \( Q \) such that \( MQ \) is in row-echelon form.

Lemma 3.24. For any linear map \( L : \mathbb{V}^m \to \mathbb{W}^n \), there exist bases \( E = (e_1, \ldots, e_m) \) of \( \mathbb{V} \) and \( F = (f_1, \ldots, f_n) \) of \( \mathbb{W} \) such that the entries of the matrix \( M = I_{E,F}(L) = I_F^{-1} \circ L \circ I_E \) are

\[
M^i_j = \begin{cases} 
\delta^i_j & \text{if } 1 \leq i, j \leq \operatorname{rank} M \\
0 & \text{, otherwise.}
\end{cases}
\]
3.7 Change of basis

Let $E$ and $F$ be bases of a vector space $\mathbb{V}^m$. Therefore, there exists a matrix

$$R = \begin{bmatrix}
R_{11} & \cdots & R_{1m} \\
\vdots & & \vdots \\
R_{m1} & \cdots & R_{mm}
\end{bmatrix}$$

such that, for each $1 \leq j \leq m$,

$$f_j = e_i R_{ij}^i.$$

Equivalently, if $E$ and $F$ are treated as row matrices of vectors, then

$$F = ER.$$

Using this notation, it is easy to derive the change of basis formulas for a vector. For each $v \in \mathbb{V}^m$, there exist coefficients $A, B \in \mathbb{R}^m$, viewed as column vectors, such that

$$v = EA = FB = FRB.$$

Since the coefficient matrices $A$ and $B$ are uniquely determined by the bases and the vector $v$, it follows that

$$A = RB.$$
Chapter 4

Affine Spaces and Maps

4.1 1-dimensional spaces

4.1.1 The real line

The real line is the 1-dimensional vector space but labeled by real numbers with respect to a chosen set of units.

4.1.2 Abstract 1-dimensional vector space

An abstract 1-dimensional vector space $L$ can be thought of as the real line with all of the numbers, except 0, erased. It is, in particular, the set of all oriented line segments in a Euclidean line that start at zero. The concepts of congruence and rescaling of vectors can still be defined geometrically. They can in turn be used to define concepts of vector addition and rescaling of a vector that satisfy properties listed in Definition 3.1.

A vector $v$ looks like this in $L$:

4.1.3 The affine line

The affine line $A$ can be viewed as an abstract 1-dimensional vector space with the origin erased. More precisely, it is the Euclidean line that satisfies the axioms in Subection 2.1.1. Associated with each point $p \in A$ is the vector space of all oriented line segments that start at $p$. We shall denote this vector space by $L_p$. 
Given two points \( p, q \in A \), there is a vector \( v \in \mathbb{L}_p \) from \( p \) to \( q \).

We denote \( v = q - p \in \mathbb{L}_p \). Given any two points \( p, q \in A \), there is a natural linear isomorphism \( I_{pq} : L_p \to L_q \), where, given any vector (i.e., oriented line segment) \( v \in L_p \), \( I_{pq}(v) \in L_q \) is the unique vector (i.e., oriented line segment) that is congruent to \( v \). Since \( I_{qr} = I_{pq} \circ I_{qr} \) and \( I_{pp} \) is the identity map, there is an abstract vector space \( \mathbb{L} \) and linear isomorphisms \( I_p : \mathbb{L} \to \mathbb{L}_p \) such that, for any \( p, q \in A \), \( I_{pq} = I_q \circ I_p^{-1} \).

### 4.2 2-dimensional spaces

#### 4.2.1 \( \mathbb{R}^2 \)

4.2.2 2-dimensional vector space
4.3. Affine spaces

4.3.1 Definition

**Definition 4.1.** An affine space associated with an \( m \)-dimensional vector space \( \mathbb{V}^m \) is a set \( \mathbb{A}^m \) with operations denoted, for any \( p, q \in \mathbb{A}^m \) and \( v \in \mathbb{V}^m \),

\[
q - p \in \mathbb{V}^m \\
p + v \in \mathbb{A}^m 
\]

which satisfy the following properties:

\[
q - p = 0 \iff q = p \\
p + v = q \iff v = q - p \\
p + (v_1 + v_2) = (p + v_1) + v_2.
\]

We will often call \( \mathbb{V}^m \) the *tangent space* of \( \mathbb{A}^m \). As we will see, it can also be viewed as the set of all possible velocity vectors of parameterized curves in \( \mathbb{A}^m \).

In particular, given any \( p \in \mathbb{A}^m \), the map

\[
I_p : \mathbb{V}^m \rightarrow \mathbb{A}^m \\
v \mapsto p + v
\]

is bijective, and its inverse map is

\[
I_p^{-1} : \mathbb{A}^m \rightarrow \mathbb{V}^m \\
q \mapsto q - p.
\]

For each \( p, q \in \mathbb{A}^m \), the vector \( I_p(q) \) is called the *position vector* of \( q \) relative to \( p \).

**Definition 4.2.** The *dimension* of an affine space \( \mathbb{A}^m \) is defined to be the dimension of the vector space associated to it.
4.3.2 Examples

The fundamental example of an affine space is Cartesian space $\mathbb{R}^m$. To distinguish between vectors and points in $\mathbb{R}^m$, we will use the following notation for the Cartesian space of points:

$$\tilde{\mathbb{R}}^m = \{ \tilde{p} = (p^1, \ldots, p^m) : p^1, \ldots, p^m \in \mathbb{R} \}.$$ 

Since this is an affine space, the rules of algebra on $\tilde{\mathbb{R}}^m$ are different from the usual ones for $\mathbb{R}^m$. They are instead the following: You cannot add two points $p_1, p_2 \in \tilde{\mathbb{R}}^m$, and you cannot rescale a point by a factor $c \in \mathbb{R}$. You can, however, do the following: Given

$$\tilde{p} = (p^1, \ldots, p^m) \in \tilde{\mathbb{R}}^m$$
$$\tilde{p}_1 = (p_1^1, \ldots, p_1^m) \in \tilde{\mathbb{R}}^m$$
$$\tilde{p}_2 = (p_2^1, \ldots, p_2^m) \in \tilde{\mathbb{R}}^m$$
$$\tilde{v} = (v^1, \ldots, v^m) \in \tilde{\mathbb{R}}^m$$

the following algebraic operations are allowed:

$$\tilde{p}_2 - \tilde{p}_1 = (p_2^1 - p_1^1, \ldots, p_2^m - p_1^m) \in \tilde{\mathbb{R}}^m$$
$$\tilde{p} + \tilde{v} = (p^1 + v^1, \ldots, p^m + v^m) \in \tilde{\mathbb{R}}^m.$$ 

With this notation, $\tilde{\mathbb{R}}^m$ is an affine space with tangent space $\tilde{\mathbb{R}}^m$.

A similar example is a vector space $V^m$ with itself as its tangent space. Another example is

$$H^m = \{ x^{m+1} = 1 \} \subset \tilde{\mathbb{R}}^{m+1},$$

with tangent space

$$T = \{ x^{m+1} = 0 \} \subset \tilde{\mathbb{R}}^{m+1}.$$ 

More generally, given $a_1, \ldots, a_{m+1}, b \in \mathbb{R}$,

$$A^m = \{ a_1 x^1 + \cdots + a_{m+1} x^{m+1} = b \} \subset \tilde{\mathbb{R}}^{m+1}.$$ 

is an affine space with tangent space

$$V^m = \{ a_1 v^1 + \cdots + a_{m+1} v^{m+1} = 0 \} \subset \tilde{\mathbb{R}}^{m+1}.$$ 

The abstract version of this is the following: Given an abstract vector space $\mathbb{W}^{m+1}$ and a nonzero linear function $\ell : \mathbb{W}^{m+1} \to \mathbb{R}$, the space

$$A^m = \{ w \in \mathbb{W}^{m+1} : \ell(w) = 1 \}$$

is affine with tangent space

$$V^m = \{ v \in \mathbb{W}^{m+1} : \ell(v) = 0 \}. $$
4.4 Affine maps

4.4.1 Affine maps of Cartesian spaces

A map $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is called affine, if there is a $n$-by-$m$ matrix $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $b \in \mathbb{R}^n$ such that, for any $\tilde{p} \in \mathbb{R}^m$,

$$F(\tilde{p}) = L(\tilde{p} - \tilde{0}) + \tilde{b}.$$  

Note that, for any $\tilde{v} \in \mathbb{R}^m$,

$$L(\tilde{v}) = F(\tilde{0} + \tilde{v}) - F(\tilde{0}).$$

More generally, for any $\tilde{p} \in \mathbb{R}^m$ and $v$ in $\mathbb{R}^m$,

$$L(\tilde{v}) = F(\tilde{p} + \tilde{v}) - F(\tilde{p}).$$

Equivalently, for any $\tilde{p}, \tilde{q} \in \mathbb{R}^m$,

$$F(\tilde{q}) = F(\tilde{p}) + L(\tilde{q} - \tilde{p}).$$

In particular, if $m = n = 1$, then $M : \mathbb{R} \rightarrow \mathbb{R}$ looks like

$$F(x) = ax + b, \; \forall x \in \mathbb{R},$$

where $a$ and $b$ are constants.

4.4.2 Abstract affine maps

Definition 4.3. Let $A^m$ and $B^n$ be affine spaces with corresponding tangent spaces $V^m$ and $W^n$. A map $M : A^m \rightarrow B^n$ is affine, if there is a linear map $L : V^m \rightarrow W^n$ such that, for any $p, q \in A^m$,

$$M(q) = M(p) + L(q - p) \quad (4.1)$$

This implies that, given $p \in A^m$, any affine map $M : A^m \rightarrow B^n$ is uniquely determined by the linear map $L : V^m \rightarrow W^n$ and $M(p) \in W^n$ and vice versa.

A affine map is a affine isomorphism, if it is both injective and surjective. An affine map $M : A^m \rightarrow \mathbb{R}$ is called an affine function.

4.4.3 Affine independence and basis

Definition 4.4. A set of points $p_0, \ldots, p_k \in A^m$ is in general position or, equivalently, are affinely independent, if the points $p_1 - p_0, \ldots, p_k - p_0 \in V^m$ are linearly independent.

Lemma 4.5. A set of points $p_0, \ldots, p_k \in A^m$ are in general position if and only if, for each $0 \leq i \leq k$, the set

$$\{p_0 - p_i, \ldots, p_k - p_i\} \subset V^m$$

consists of the zero vector and a linearly independent subset of $V^m$. 
Definition 4.6. An ordered list of points, \((p_0, \ldots, p_m)\), is an affine basis of \(A^m\), if the vectors \((p_1 - p_0, \ldots, p_m - p_0)\) is a basis of \(V^m\).

Lemma 4.7. Given an affine basis \(P = (p_0, \ldots, p_n) \subset A^m\), the map

\[
I_P : \mathbb{R}^m \to A^m \\
(a^1, \ldots, a^m) \mapsto p_0 + a^1(p_1 - p_0) + \cdots + a^m(p_m - p_0)
\]

is an affine isomorphism.

Lemma 4.8. Given an affine isomorphism \(I : A^m \to B^m\), an ordered set \((p_0, \ldots, p_m)\) is an affine basis of \(A^m\) if and only if \((I(p_0), \ldots, I(p_m))\) is an affine basis of \(B^m\).

Example. An affine basis of \(\mathbb{R}^3\) is given by the points

\((0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\).

4.4.4 The space of affine maps

Let \(\text{Hom}_{aff}(A^m, B^n)\) denote the space of all affine maps from \(A^m\) to \(B^n\).

Lemma 4.9. \(\text{Hom}_{aff}(A^m, B^n)\) is an affine space with tangent space \(\text{Hom}_{aff}(V^m, W^n)\).

Equation (4.1) implies that the map

\[
I_p : \text{Hom}(V^m, W^n) \times B^n \to \text{Hom}_{aff}(A^m, B^n) \\
(L, b) \mapsto M,
\]

where, for each \(q \in A^m\),

\[M(q) = L(q - p) + b,
\]

is an bijection.

Given affine bases \(P = (p_0, \ldots, p_m)\) of \(A^m\) and \(Q = (q_0, \ldots, q_n)\) of \(B^n\), let

\[I_{P,Q} : \text{Hom}_{aff}(\mathbb{R}^m, \mathbb{R}^n) \to \text{Hom}(A^m, B^n),
\]

denote the map, where for each \(M \in \text{Hom}(\mathbb{R}^m, \mathbb{R}^n)\),

\[I_{P,Q}(M) = I_Q \circ M \circ I_P^{-1}.
\]

Lemma 4.10. The map \(I_{P,Q}\) is an affine isomorphism.

Lemma 4.11. \(\dim \text{Hom}(A^m, B^n) = (m + 1)n\).
Chapter 5
Concrete Category Theory

The concept of a category is implicit in much of the mathematics you have already learned. The idea is that, within a specific area such as linear algebra, you often restrict your attention to a certain class of sets with given properties, such as vector spaces, and a corresponding class of maps, such as linear transformations.

Another example is the class of open sets in $\mathbb{R}^n$ for any $n > 0$ and the corresponding class of smooth maps from an open set in $\mathbb{R}^n$ to $\mathbb{R}^m$ for any $m, n > 0$.

A functor is in a loose sense a map from one category to another. We explain more precisely below.

5.1 Definition of a category

A category $\mathcal{C}$ consists of the following:

- A collection $\mathcal{S}$ of things we call objects
- A collection $\mathcal{M}$ of things we call morphisms or arrows. Associated with any morphism are two objects, one we call the source and the other the target. Any morphism can be written as $f : A \rightarrow B$, where $F$ is the name of the morphism, $A$ is the source, and $B$ is the target.

Moreover, the following properties must hold:

- (Composition) Any two morphisms $f : A \rightarrow B$, $g : B \rightarrow C$ (where the target of $f$ is the source of $g$) uniquely determine a morphism with source $A$ and target $C$, which we denote by $g \circ f : A \rightarrow C$.

- (Associativity) Given morphisms $f : A \rightarrow B$, $g : B \rightarrow C$, and $h : C \rightarrow D$,

  $$(h \circ g) \circ f = h \circ (g \circ f) : A \rightarrow D.$$  

- (Identity) For each object $A$, there is a morphism $I_A$ such that, for any morphism $f : A \rightarrow B$, $f \circ I_A = f$ and, for any morphism $g : B \rightarrow A$, $I_A \circ g = g$. 

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Here, we will restrict to the following types of categories: Objects are sets that satisfy a common set of properties. In this situation an object is often called a space. A morphism is a map between two spaces, where the source is the domain and the target is the range, that satisfies a specified set of properties. The map $g \circ f$ is the composition of the two maps, and $I_A : A \to A$ is simply the identity map.

## 5.2 Examples

### 5.2.1 Sets

Let $\text{Set}$ denote the category, where the spaces are sets and the maps are maps from a set to another.

### 5.2.2 Finite dimensional real vector spaces

Let $\text{Lin}$ denote the category, where the spaces are finite dimensional real vector spaces and the maps are linear maps between two vector spaces.

### 5.2.3 Finite dimensional affine spaces

Let $\text{Aff}$ denote the category, where the spaces are finite dimensional real affine spaces and the maps are affine maps between two affine spaces.

### 5.2.4 $C^1$ manifolds

Let $\text{Man}$ denote the category, where the spaces are open subsets of a finite-dimensional affine space, and the maps are $C^1$ maps from one space to another. Later, this category will be extended to a larger collection of spaces and maps.

## 5.3 Definition of Functor

A **covariant functor** $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ consists of maps

\[ \mathcal{F}_S : S_1 \to S_2 \]
\[ \mathcal{F}_M : M_1 \to M_2, \]

such that the following hold:

- For each map $f : X \to Y$ in $\mathcal{M}_1$, the corresponding map is
  \[ \mathcal{F}_M(f) : \mathcal{F}_S(X) \to \mathcal{F}_S(Y). \]

- Given maps $f : X \to Y$ and $g : Y \to Z$,
  \[ \mathcal{F}_M(g \circ f) = \mathcal{F}_M(g) \circ \mathcal{F}_M(f) \]
5.4. Examples of functors

5.4.1 Forgetful functors

Given any of the categories $\mathcal{C}$ above, there is a forgetful functor $\mathcal{F} : \mathcal{C} \rightarrow \text{Set}$, since each space in $\mathcal{C}$ is a space in $\text{Set}$ and each map in $\mathcal{C}$ is a map in $\text{Set}$.

Another forgetful functor $\mathcal{F} : \text{Lin} \rightarrow \text{Aff}$ forgets the origin of each vector space. In other words, any vector space is itself an affine space and any linear map is an affine map.

There is also the forgetful functor $\mathcal{F} : \text{Aff} \rightarrow \text{Man}$ that forgets the affine structure but remembers the topological structure of each affine space. Each affine space is itself an open space of itself, and each map in $\text{Aff}$ is a $C^1$ map.

5.4.2 Dualization

There is a contravariant functor $\text{Dual} : \text{Lin} \rightarrow \text{Lin}$, where, for each vector space $V$, $\text{Dual}(V) = V^*$ and, given $A : V \rightarrow W$, $\text{Dual}(A) = A^t : W^* \rightarrow V^*$. 

- For each space $X$, 
  \[ \mathcal{F}_M(\text{id}_X) = \text{id}_{\mathcal{F}_S(X)}. \]

A contravariant functor $\mathcal{F} : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ reverses the arrows. It consists of maps

\[ \mathcal{F}_S : S_1 \rightarrow S_2 \]
\[ \mathcal{F}_M : M_1 \rightarrow M_2, \]

such that the following hold:

- For each map $f : X \rightarrow Y$ in $\mathcal{M}_1$, the corresponding map is 
  \[ \mathcal{F}_M(f) : \mathcal{F}_S(Y) \rightarrow \mathcal{F}_S(X). \]

- Given maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, 
  \[ \mathcal{F}_M(g \circ f) = \mathcal{F}_M(f) \circ \mathcal{F}_M(g) \]

- For each space $X$, 
  \[ \mathcal{F}_M(\text{id}_X) = \text{id}_{\mathcal{F}_S(X)}. \]
Chapter 6

Euclidean space

The concepts of a vector space and an affine space provide an abstract algebraic formulation of the properties of points and lines in Euclidean space. In this chapter we introduce an abstract version of the dot product, which is then used to define the length of a vector and the angle between two vectors in an abstract vector space. This in turn can be used to define the distance between two points in an affine space and the angle defined by three points.

6.1 Cartesian space

6.1.1 Length and angle using the dot product

Recall that the dot product of two vectors in $\mathbb{R}^m$ is defined to be

$$\langle v_1^1, \ldots, v_1^m \rangle \cdot \langle v_2^1, \ldots, v_2^m \rangle = v_1^1 v_2^1 + \cdots + v_1^m v_2^m.$$  \hspace{1cm} (6.1)

The length or magnitude of a vector is

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}.$$

Recall that, if the angle between two nonzero vectors $v_1$ and $v_2$ is $\theta$, where $0 \leq \theta \leq \pi$, then

$$\cos \theta = \frac{v_1 \cdot v_2}{|v_1||v_2|}.$$

In particular, the two vectors are orthogonal, if

$$\vec{v}_1 \cdot \vec{v}_2 = 0,$$

and, if this holds,

$$|\vec{v}_1|^2 + |\vec{v}_2|^2 = |\vec{v}_2 - \vec{v}_1|^2,$$

which is the Pythagorean theorem.
CHAPTER 6. EUCLIDEAN SPACE

The dot product also has the following fundamental properties: For each \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^m \) and \( a \in \mathbb{R} \),

\[
\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_2 \cdot \vec{v}_1 \\
(\vec{v}_1 + \vec{v}_2) \cdot \vec{v}_3 = \vec{v}_1 \cdot \vec{v}_3 + \vec{v}_2 \cdot \vec{v}_3 \\
(a \vec{v}_1) \cdot \vec{v}_2 = a(\vec{v}_1 \cdot \vec{v}_2) \\
\vec{v}_1 \cdot \vec{v}_1 \geq 0 \\
\vec{v}_1 \cdot \vec{v}_1 = 0 \iff \vec{v}_1 = 0.
\]

6.2 Inner product

It turns out that almost all geometric theorems about Euclidean space can be proved using only the properties of the dot product, and its definition (6.1) is rarely needed. Due to this, it turns out to be useful to define an abstract concept of the dot product, which is usually called the inner product.

**Definition 6.1.** An *inner product* on a vector space \( V \) is a real-valued function of two vectors, which we denote by \( \langle v_1, v_2 \rangle \in V \), that satisfies the following properties for any \( v_1, v_2, v_3, v \in V \) and \( a \in \mathbb{R} \):

\[
\begin{align*}
\langle v_1, v_2 \rangle &= v_2 \cdot v_1 \\
\langle v_1 + v_2, v_3 \rangle &= v_1 \cdot v_3 + v_2 \cdot v_3 \\
\langle av_1, v_2 \rangle &= a \langle v_1, v_2 \rangle \\
v \cdot v &\geq 0 \\
v \cdot v = 0 \iff v = 0.
\end{align*}
\]

Given an inner product on \( V \), we define the *norm* or *magnitude* of a vector \( v \in V \) to be

\[
|v| = \sqrt{v \cdot v}. \tag{6.2}
\]

An *inner product space* is a vector space with an inner product. Cartesian space \( \mathbb{R}^m \) with the dot product is an inner product space. There are, however, an infinite number of different possible inner products on a vector space \( V \).

**Lemma 6.2.** Given a basis \( (e_1, \ldots, e_m) \) of a vector space \( \mathbb{V}^m \), an inner product on \( \mathbb{V}^m \) is uniquely determined by the symmetric matrix

\[
A = \begin{bmatrix}
A_{11} & \cdots & A_{1m} \\
\vdots & \ddots & \vdots \\
A_{m1} & \cdots & A_{mm}
\end{bmatrix},
\]

where \( A_{ij} = e_i \cdot e_j \). In particular, given vectors \( v = v^i e_i \) and \( w = w^i e_i \),

\[
v \cdot w = \begin{bmatrix} v^1 & \cdots & v^m \end{bmatrix} A \begin{bmatrix} w^1 \\
\vdots \\
w^m \end{bmatrix}. \tag{6.3}
\]
6.3. NORM

Given a symmetric matrix $A$, (6.3) does not necessarily define an inner product. Simple counterexamples for $m = 2$ include

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

**Definition 6.3.** A symmetric $m$-by-$m$ matrix $A$ is **positive definite**, if (6.3) defines an inner product.

**Lemma 6.4.** A symmetric 2-by-2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{12} = a_{21}$, defines an inner product if and only if

$$a_{11}, a_{22} > 0 \quad \text{and} \quad a_{11}a_{22} - a_{12}^2 > 0.$$

### 6.3 Norm

The norm defined, using an inner product, by (6.2), satisfies the following basic properties:

For any $v, v_1, v_2 \in \mathbb{V}$,

$$|v| \geq 0,$$

$$|v| = 0 \iff v = 0,$$

$$|cv| = c|v|,$$

$$|v_1 + v_2| \leq |v_1| + |v_2|.$$ 

There are, however, other norms that can be defined on a vector space and satisfy the properties above. Examples of norms on $\mathbb{R}^m$ are:

$$|\langle v^1, \ldots, v^m \rangle|_1 = |v^1| + \cdots + |v^m|,$$

$$|\langle v^1, \ldots, v^m \rangle|_p = \left( \sum_{i=1}^{m} |v^i|^p \right)^{1/p}, \text{ where } 1 \leq p < \infty,$$

$$|\langle v^1, \ldots, v^m \rangle|_{\infty} = \max(|v^1|, \ldots, |v^m|).$$

### 6.4 Orthogonal maps

**Definition 6.5.** An **orthogonal map** is a linear map $F : \mathbb{V}^m \to \mathbb{W}^n$, where $\mathbb{V}^m$ and $\mathbb{W}^n$ are inner product spaces, such that

$$F(v_1) \cdot F(v_2) = v_1 \cdot v_2, \quad \forall \ v_1, v_2 \in \mathbb{V}^m.$$ 

If $\mathbb{W}^n = \mathbb{V}^m$, then $F$ is also called an orthogonal transformation.

Let $O(\mathbb{V}^m, \mathbb{W}^n)$ denote the set of all orthogonal maps from $\mathbb{V}^m$ to $\mathbb{W}^n$ and $O(\mathbb{V}^m) = O(\mathbb{V}^m, \mathbb{V}^m)$. 

6.5 Orthonormal vectors

Let \( V^m \) be \( m \)-dimensional inner product space.

**Definition 6.6.** A set of vectors, 
\[
E = \{e_1, \ldots, e_k\} \subset V^m
\]
is *orthonormal*, if for any \( 1 \leq i, j \leq k \),
\[
e_i \cdot e_j = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

Note that if \( v \in [E] \), then
\[
v = \sum_{i=1}^{k} a^i e_i,
\]
where, for each \( 1 \leq i \leq k \),
\[
a^i = v \cdot e_i.
\]

**Lemma 6.7.** A set of orthonormal vectors is linearly independent.

For each orthonormal set \( E \), define 
\[
[E]^\perp = \{v \in V^m : v \cdot e = 0, \ \forall e \in [E]\}
\]
and the linear maps
\[
\pi_E : V^m \to [E] \\
v \mapsto \sum_{i=1}^{k} (v \cdot e_i)e_i
\]
\[
\pi_E^\perp : V^m \to V^m \\
v \mapsto v - \pi_E(v).
\]

**Lemma 6.8.** For each \( v \in V^m \),
\[
v = v_1 + v_2, \text{ where } v_1 \in [E] \text{ and } v_2 \in [E]^\perp
\]
if and only if
\[
v_1 = \pi_E(v) \text{ and } v_2 = \pi_E^\perp(v).
\]

**Lemma 6.9.** A finite-dimensional inner product space has at least one orthonormal basis.

**Lemma 6.10.** If \( E = (e_1, \ldots, e_m) \) is an orthonormal basis of \( V^m \), then \( F = (f_1, \ldots, f_m) \) is also an orthonormal basis if and only if
\[
f_j = e_i M^i_j, \ \forall \ 1 \leq j \leq m,
\]

where \( M^i_j \) are the matrix entries.
where
\[ M_j^i M_k^i = \delta_{jk}, \ \forall \ 1 \leq j, k \leq m, \]
or, equivalently, if and only if
\[ F = EM, \]
where
\[ M^t M = I. \] (6.4)

Let \( O(m) \) denote the set of all \( m \)-by-\( m \) matrices satisfying (6.4). It is a group under matrix multiplication.

**Lemma 6.11.** Given an orthonormal basis \( E \) of an inner product space \( V^m \), there is a group homomorphism

\[ O(m) \to O(V^m) \]

\[ M \mapsto A, \]

where, for each \( v = e_i a^i \),

\[ A(e_i a^i) = e_j M_j^i a^i. \]

### 6.6 Metric spaces

Recall that the distance between two points \( p_1, p_2 \in \mathbb{R}^m \) is given by

\[ d(p_1, p_2) = |p_2 - p_1| = \sqrt{(p_2 - p_1) \cdot (p_2 - p_1)}. \]

and satisfies the following properties for any \( p_1, p_2, p_3 \in \mathbb{R}^m \):

\[ d(p_1, p_2) = d(p_2, p_1) \]
\[ d(p_1, p_2) \geq 0 \]
\[ d(p_1, p_2) = 0 \iff p_2 = p_1 \]
\[ d(p_1, p_3) \leq d(p_1, p_2) + d(p_2, p_3). \] (6.5)

The leads to the following abstract definition:

**Definition 6.12.** A **metric space** is a set \( X \) and a function \( d : X \times X \to [0, \infty) \) that satisfies, for every \( p_1, p_2, p_3 \in X \), the properties given by (6.5).

If \( X \) and \( Y \) are both metric spaces, then a map \( F : X \to Y \) is called an **isometry**, if for any \( x_1, x_2 \in X \),

\[ d_Y(F(x_1), F(x_2)) = d_X(x_1, x_2). \]

If \( A \) is an affine space with tangent space \( V \) and \(| \cdot |\) is a norm on \( V \), then \( A \) with the distance function

\[ d(p_1, p_2) = |p_2 - p_1| \]

is a metric space.
6.7 Isometries of Euclidean space

Definition 6.13. An \( m \)-dimensional Euclidean space is an \( m \)-dimensional affine space whose tangent space is an inner product space.

Lemma 6.14. Let \( \mathbb{A}^m \) and \( \mathbb{B}^n \) be Euclidean spaces. A map \( F : \mathbb{A}^m \to \mathbb{B}^n \) is an isometry, if and only if there exists \( L \in O(\mathbb{V}^m, \mathbb{W}^n) \) such that the following holds: For any \( p \in \mathbb{A}^m \), there exists \( q \in \mathbb{B}^n \) such that, for any \( a \in \mathbb{A}^m \):

\[
F(a) = L(a - p) + q.
\]

Definition 6.15. An isometry \( F : \mathbb{A}^m \to \mathbb{A}^m \) is called a rigid motion.

Lemma 6.16. The set \( G \) of all possible rigid motions of \( m \)-dimensional Euclidean space is a group, because it satisfies the following properties:

1. If \( F_1, F_2 \in G \), then \( F_2 \circ F_1 \in G \).
2. The identity map \( I : \mathbb{A}^m \to \mathbb{A}^m \) is in \( G \).
3. Any rigid motion \( F : \mathbb{A}^m \to \mathbb{A}^m \) has an inverse map \( F^{-1} : \mathbb{A}^m \to \mathbb{A}^m \) that is also a rigid motion.
Chapter 7

Curves in Affine Space

7.0.1 Curves in \( \mathbb{R}^m \)

A parameterized curve in \( \mathbb{R}^m \) is a map of the form

\[
c : I \rightarrow \mathbb{R}^m
\]

\[
t \mapsto c(t) = (c^1(t), \ldots, c^m(t)),
\]

where \( I \subset \mathbb{R} \) is an interval and each \( c^i \) is a real-valued function on \( I \).

**Definition 7.1.** The curve \( c \) is defined to be *continuous*, if each of the functions \( c^1, \ldots, c^m \) are continuous.

The curve \( c \) is defined to be \( C^1 \), if the functions \( c^1, \ldots, c^m \) and their derivatives \( \dot{c}^1, \ldots, \dot{c}^m \) are continuous.

The curve \( c \) is \( C^2 \), if, in addition, the second derivatives \( \ddot{c}^1, \ldots, \ddot{c}^m \) are continuous.

Since, given \( s, t \in I \) with \( s \neq t \),

\[
\frac{c(s) - c(t)}{s - t} \in \mathbb{R}^m,
\]

it follows that if \( c \) is \( C^1 \), then for each \( t \in I \),

\[
\dot{c}(t) = \lim_{s \to t} \frac{c(s) - c(t)}{s - t}
\]

\[
= \lim_{s \to t} \left\langle \frac{c^1(s) - c^1(t)}{s - t}, \ldots, \frac{c^m(s) - c^m(t)}{s - t} \right\rangle
\]

\[
= \langle \dot{c}^1(t), \ldots, \dot{c}^m(t) \rangle \in \mathbb{R}^m.
\]

Therefore, the derivative \( \dot{c} \) is a map from \( I \) to \( \mathbb{R}^m \). Similarly, if \( c \) is \( C^2 \), then \( \ddot{c} \) is also a map from \( I \) to \( \mathbb{R}^m \).
7.0.2 Curves in affine space

A parameterized curve in \( \mathbb{A}^m \) is a map of the form
\[ c : I \to \mathbb{A}^m, \]
where \( I \subset \mathbb{R} \) is an interval. The curve \( c \) is defined to be continuous, if, for any \( p \in \mathbb{A}^m \), the curve \( I_p^{-1} \circ c : \mathbb{R} \to \mathbb{R}^m \) is continuous. The curve \( c \) is \( C^1 \) if \( I_p^{-1} \circ c \) is \( C^1 \) and \( C^2 \) if \( I_p^{-1} \circ c \) is \( C^2 \).

Lemma 7.2. Given a curve \( c : I \to \mathbb{A}^m \) and \( pq \in \mathbb{A}^m \), the curve \( I_p^{-1} \circ c : I \to \mathbb{R}^m \) is continuous if and only if \( I_q^{-1} \circ c \) is continuous. Similar statements hold for \( C^1 \) and \( C^2 \).

7.1 Velocity and acceleration

Definition 7.3. The velocity of a \( C^1 \) curve \( c : I \to \mathbb{A}^m \) at time \( t_0 \in I \) is defined to be the derivative of \( c \) at time \( t_0 \),
\[ v(t) = \dot{c}(t) = \lim_{s \to t} \frac{c(s) - c(t)}{s - t} \in \mathbb{V}^m. \]

Since \( c'(t) \in \mathbb{V}^m \) is tangent to \( c(t) \in \mathbb{A}^m \), the vector space \( \mathbb{V}^m \) is called the tangent space of \( \mathbb{A}^m \).

If \( c \) is \( C^2 \), then the acceleration is the second derivative of \( c \),
\[ a(t) = \ddot{c}(t) = \lim_{s \to t} \frac{c'(s) - c'(t)}{s - t}. \]

7.2 Curves in Euclidean space

Let \( \mathbb{E}^m \) be \( m \)-dimensional Euclidean space with tangent space \( \mathbb{E}^m \).

7.2.1 Velocity, speed, and length

Given a \( C^1 \) curve \( c : [t_0, t_1] \to \mathbb{E}^m \) with velocity \( v : I \to \mathbb{E}^m \), its speed is defined to be \( \sigma = |v| : I \to [0, \infty) \). The distance along the curve, i.e., the length of the curve, between two points \( c(t_0) \) and \( c(t_1) \) is therefore
\[ \ell = \int_{t_0}^{t_1} \sigma(t) \, dt. \]

In particular, we can define the arclength function to be
\[ [t_0, t_1] \to [0, \ell] \]
\[ t \mapsto s(t) = \int_{\tau=t_0}^{\tau=t} \sigma(\tau) \, d\tau. \]
If the velocity is never zero, then \( \sigma \) is continuous and, by the fundamental theorem of calculus, \( s' = \sigma \). Moreover, \( s \) is strictly increasing, and therefore, by the chain rule, has a \( C^1 \) inverse function \( t(s) \). The **arc length parameterization** of \( c \) is defined to be

\[
\hat{c} : [0, \ell] \to \mathbb{E}^m
\]

\[
s \mapsto c(t(s)).
\]

Since the speed of \( \hat{c} \) is 1, it is called a *unit speed curve*.

A simple fact we will use is the following: Given any \( C^1 \) function \( f : [t_0, t_1] \to \mathbb{R} \),

\[
\frac{d}{ds} (f(t(s))) = \frac{f'(t(s))}{s'(t)}.
\]

### 7.2.2 Acceleration and curvature

The velocity of a \( C^2 \) curve \( c : I \to \mathbb{E}^m \) can be written as

\[
v(t) = \sigma(t)u(t), \text{ where } u = \frac{v}{|v|} \text{ is a unit vector for each } t \in I.
\]

Therefore, the acceleration of \( c \) is

\[
a = \dot{v} = \sigma'u + \sigma u'.
\]

The first term represents the acceleration of speed along the curve, while the second represents the acceleration due to the change in direction of the curve.

We now use the following lemma:

**Lemma 7.4.** If \( u : I \to T_m \) is \( C^1 \) and \( u(t) \) is a unit vector for every \( t \in I \), then \( u'(t) \) is orthogonal to \( u(t) \), for every \( t \in I \).

**Proof.**

\[
0 = \frac{d}{dt} (u \cdot u) = 2u \cdot u'.
\]

Here, this implies that the acceleration due to change in direction is orthogonal to the tangent direction.

### 7.2.3 Curvature

Geometric properties of the curve should not depend on the parameterization. We therefore want properties that do not depend on the speed or acceleration along the curve. It suffices to study only unit speed curves. Notice that this restricts us to curves that have at least one parameterization with nonzero velocity everywhere along the curve.

The other thing to note is that a curve is straight if \( u \), as defined above, never changes direction. Moreover, the greater the magnitude of \( u' \) is, the more sharply curved the curve is. It therefore makes sense to define the curvature at each point \( c(t) \) of a unit speed \( C^2 \) curve to be

\[
\kappa(t) = |a(t)| = |u'(t)|.
\]
7.2.4 Frenet-Serret frame in Euclidean 2-space

Let \( c : [0, T] \to \mathbb{E}^2 \) be a \( C^2 \) curve with nonzero velocity \( \cdot c \) and speed \( \sigma = |\cdot c| \). Let \( e_1 \) be the unit vector pointing in the same direction as \( \dot{c} \), i.e.,

\[
\dot{c} = |\dot{c}|e_1 = \sigma e_1.
\]

There exists, for each \( t \in [0, T] \), a unit vector \( e_2(t) \), that is a \( C^1 \) function of \( t \), such that

\[
E(t) = [e_1(t) \ e_2(t)]
\]

is an orthonormal basis of \( \dot{E}^2 \). By Lemma 7.4, \( e_1 \cdot e_1' = 0 \). This implies that \( e_1'(t) \) is a scalar multiple of \( e_2 \). It follows that there is a function \( \kappa \) such that, for each \( t \in [0, T] \),

\[
e_1'(t) = \sigma(t)\kappa(t)e_2(t).
\]

Again, since \( e_2' \) is orthogonal to \( e_2 \), it is a scalar multiple of \( e_1 \). On other hand, since \( e_1 \cdot e_2 = 0 \),

\[
0 = \frac{d}{dt}(e_1 \cdot e_2) = e_1' \cdot e_2 + e_1 \cdot e_2' = \sigma \kappa + e_1 \cdot e_2',
\]

and therefore

\[
e_1 \cdot e_2' = -\sigma \kappa.
\]

It follows that

\[
e_2' = -\sigma \kappa e_1.
\]

The 1-parameter family \( E(t) \) of orthonormal bases of \( \dot{E}^2 \) is called the Frenet-Serret frame of \( c \). We have shown that it satisfies the equations

\[
\begin{bmatrix}
e_1'(t) \\
e_2'(t)
\end{bmatrix} = \sigma \begin{bmatrix}
\kappa e_2(t) \\
-\kappa e_1(t)
\end{bmatrix} = \sigma \begin{bmatrix}
e_1(t) \\
e_2(t)
\end{bmatrix} \begin{bmatrix}
0 \\
\kappa(t)
\end{bmatrix}.
\]

This is known as the Frenet-Serret equations in Euclidean 2-space.

If we however, assume that \( c : [0, T] \to \mathbb{E}^2 \) is a \( C^2 \) unit speed curve such that a Frenet-Serret frame satisfying the Frenet-Serret equations (7.2), then the curvature function uniquely determines the shape but not the position of the curve.

**Theorem 7.5.** Given a continuous function \( \kappa : [0, T] \to \mathbb{E}^2 \), \( p_0 \in \mathbb{E}^2 \), and \( e_0 \in \dot{E}^2 \), there exists a unique curve \( c : [0, T] \to \mathbb{E}^2 \) and moving frame

\[
E(t) = [e_1(t) \ e_2(t)] , \quad 0 \leq t \leq T,
\]

such that \( \dot{c} = e_1 \) and \( E \) satisfies (7.2).

**Corollary 7.6.** Given a continuous function \( \kappa : [0, T] \to \mathbb{R} \), if a curve \( c_1 \) with moving frame \( E_1 \) and another curve \( c_2 \) with moving frame \( E_2 \) both satisfy \( \dot{c} = e_1 \) and (7.2), then there exists a rigid motion \( R : \mathbb{E}^2 \to \mathbb{E}^2 \) such that \( c_2 = R \circ c_1 \).
7.2. CURVES IN EUCLIDEAN SPACE

7.2.5 Frenet-Serret frame in Euclidean 3-space

The following lemma will be useful:

**Lemma 7.7.** If $v_1, v_2 : I \to \mathbb{R}^m$ are $C^1$ vector-valued functions satisfying

$$v_1 \cdot v_2 = 0,$$

then

$$v'_1 \cdot v_2 + v_1 \cdot v'_2 = 0,$$

which is equivalent to

$$v_2 \cdot v'_1 = -v_1 \cdot v'_2.$$  (7.3)

Let $c : I \to \mathbb{R}^3$ be a $C^2$ curve with nonzero velocity $\cdot c$, speed $\sigma = |\dot{c}|$, speed, and

$$e_1 = \frac{\dot{c}}{|\dot{c}|}.$$  

Also, assume that $\dot{e}_1$ is nonzero and let $e_2$ be the unit vector pointing in the direction of $\dot{e}_1$. It follows that there is a nonnegative function $\kappa$ such that

$$\dot{e}_1 = \sigma \kappa e_2.$$  (7.4)

There now exists a $C^1$ vector-valued function $e_3 : I \to \mathbb{R}^3$ such that, for each $t \in I$,

$$E(t) = [e_1(t) \ e_2(t) \ e_3(t)]$$

is an orthonormal basis of $\mathbb{R}^3$. Since $e_3 \cdot \dot{e}_3 = 0$, $\dot{e}_3$ is a linear combination of $e_1$ and $e_2$. By Lemma 7.7 and (7.4),

$$e_1 \cdot \dot{e}_3 = -e_3 \cdot \dot{e}_1 = 0.$$  

It follows that $\dot{e}_3$ is a scalar multiple of $e_2$. In other words, there is a function $\tau$ such that

$$\dot{e}_3 = -\sigma \tau e_2.$$  

From the equations above and Lemma 7.7,

$$e_1 \cdot e'_2 = -e_2 \cdot e'_1 = -\kappa \quad \text{and} \quad e_3 \cdot e'_2 = -e_2 \cdot e'_3 = \tau,$$

and therefore

$$\dot{e}_2 = -\kappa e_1 + \tau e_3.$$  

Everything above can be summarized as following: Let $c : I \to \mathbb{R}^3$ be a $C^2$ curve, where its velocity $v(t)$ and accelerations $a(t)$ are linearly independent for each $t \in I$. Then there exists, for each $t \in I$, an orthonormal frame $E(t)$ such that

$$[\dot{e}_1(t) \ \dot{e}_2(t) \ \dot{e}_3(t)] = \sigma [\kappa e_2(t) \ -\kappa e_1(t) + \tau e_3(t) \ -\tau e_1(t)].$$

$$= \sigma [e_1(t) \ e_2(t) \ e_3(t)] \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}.$$  (7.5)

This frame is called the Frenet-Serret frame.
Chapter 8

Surfaces in affine 3-space

8.1 Preliminaries

A set $O \subset \mathbb{R}^m$ is called open, if, for any $p \in O$, there is an $r > 0$ such that

$$B(p, r) \subset O.$$ 

Think of an open set as one that does not contain any point on its boundary. A set $D \subset \mathbb{R}^m$ is called an open domain, if it is an open set. Given a point $p \in \mathbb{R}^m$, an open neighborhood of $p$ is an open set that contains $p$. 

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Chapter 9

Surfaces in Cartesian 3-space

9.1 Informal but flawed descriptions of a surface

1. **Graph:** Given an open domain $D \subset \mathbb{R}^2$ and a $C^1$ function $h : D \to \mathbb{R}$, let

$$ S = \{ z = h(x, y) \} = \{ (x, y, h(x, y)) : (x, y) \in D \}. $$

**Example:**

$$ S = \{ z = \frac{1}{\sqrt{1 - x^2 - y^2}} \}. $$

2. **Level set:** Given an open set $O \subset \mathbb{R}^3$ and a $C^1$ function $f : O \to \mathbb{R}$, let

$$ S = \{ f(x, y, z) = 0 \} = \{ (x, y, z) : f(x, y, z) = 0 \}. $$

**Examples:**

$$ S = \{ x^2 + y^2 + z^2 - 1 = 0 \} $$

$$ S = \{ x^2 + y^2 - z^2 = 0 \}. $$

3. **Parameterized surface:** Given an open domain $D \subset \mathbb{R}^2$ and a $C^1$ map $\Phi : D \to \mathbb{R}^3$,

$$ S = \Phi(D) = \{ (\Phi(u, v) : (u, v) \in D \} = \{ (x(u, v), y(u, v), z(u, v) : (u, v) \in D \}, $$

where

$$ \Phi(u, v) = (x(u, v), y(u, v), z(u, v)). $$

**Example:**

$$ S = \{ (r \cos \theta, r \sin \theta, r) \}. $$

But these descriptions are FLAWED. And are they equivalent? How can these descriptions be made abstract for a surface in $\mathbb{A}^3$?
9.2 2-planes in 3-space

The simplest surface in \( \mathbb{R}^3 \) is a 2-plane. It can be described as follows:

1. **Graph:** Given an affine function \( h(x, y) = ax + by + c \),

let

\[
S = \{ z = h(x, y) \} = \{ z = ax + by + c \}.
\]

**Examples:**

\[
\begin{align*}
S &= \{ z = 0 \} \\
S &= \{ z = x + y + 1 \}.
\end{align*}
\]

The only shortcoming is that vertical planes are not graphs. This can be resolved by allowing \( x \) or \( y \) be the output variable and the other two variables be the input variables.

2. **Level set.** Given an affine function \( f(x, y, z) = \alpha x + \beta y + \delta z + \gamma \),

let

\[
S = \{ f(x, y, z) = 0 \} = \{ \alpha x + \beta y + \delta z + \gamma = 0 \} = \left\{ \begin{bmatrix} \alpha & \beta & \delta \\ x & y & z \end{bmatrix} + \gamma = 0 \right\}.
\]

This, however, is not always a surface, because, if \( \alpha = \beta = \delta = 0 \), then \( S \) is not a 2-plane. An extra assumption is needed, namely \( \langle \alpha, \beta, \gamma \rangle \neq 0 \). We can write this as follows: First, recall that the differential of \( f \) is defined to be

\[
\begin{align*}
&\text{df} = \partial_x f \, dx + \partial_y f \, dy + \partial_z f \, dz.
\end{align*}
\]

In order for \( S \) to be a plane and therefore a surface, we need to assume that

\[
\begin{align*}
&\text{df} \neq 0
\end{align*}
\]

3. **Parameterized surface.** Given an affine map \( \Phi : \mathbb{R}^2 \to \mathbb{R}^3 \) given by

\[
\Phi(u, v) = (x_0, y_0, z_0) + A(u, v),
\]

where \( (x_0, y_0, z_0) \in \mathbb{R}^3 \) and

\[
A = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \\ a_1^3 & a_2^3 \end{bmatrix},
\]

where

\[
\begin{align*}
&\text{df} = \partial_x f \, dx + \partial_y f \, dy + \partial_z f \, dz.
\end{align*}
\]
9.3. LOCAL DESCRIPTIONS OF A SURFACE

Let

\[ S = \{ \Phi(u, v) : (u, v) \} \]

\[ = \{ (x_0, y_0, z_0) + A(u, v), (u, v) \in \mathbb{R}^2 \} \]

\[ = \{ (x_0 + a_1^1 u + a_1^2 v, y_0 + a_1^3 u + a_2^3 v, z_0 + a_3^1 u + a_3^2 v) \}. \]

This, however, does not always work. If the rank of \( A \) is not maximal, i.e., 2, then \( S \) is either a point or a line. We must therefore assume that the rank of \( A \) is maximal. Equivalently, we must assume \( \ker A = \{0\} \).

The assumptions needed for the function \( f \) and map \( \Phi \) are nondegeneracy conditions. Notice that they can be expressed in terms of the differentials of \( f \) and \( \Phi \). In particular, the set \( S = \{ f = 0 \} \) is a surface, if

\[ df \neq 0. \]

The set \( S = \Phi(\mathbb{R}^2) \) is a surface, if the Jacobian of \( \Phi \),

\[ \partial \Phi = \begin{bmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \\ \partial_u z & \partial_v z \end{bmatrix} \]

has maximal rank.

9.3 Local descriptions of a surface

1. **Graph:** Given an open domain \( D \subset \mathbb{R}^2 \) and \( C^1 \) function \( h : D \to \mathbb{R} \), the set

\[ S = \{ z = h(x, y) \} \]

is a surface. It suffices to do this *locally*.

**Definition 9.1.** A set \( S \subset \mathbb{R}^3 \) is a *surface*, if for any \( p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3 \), there exists an open neighborhood \( O \subset \mathbb{R}^3 \) of \( p_0 \), an open domain \( D \subset \mathbb{R}^2 \), and a \( C^1 \) function \( f : D \to \mathbb{R} \) such that

\[ S \cap O = \{ z = f(x, y) \}. \]

2. **Level set:** We start with a global definition. If

\[ S = \{ f(x, y, z) = 0 \}, \]

where \( O \subset \mathbb{R}^3 \) is open and \( f : O \to \mathbb{R} \) is \( C^1 \), then \( S \) is a surface.

The equivalent local definition is the following:

**Definition 9.2.** A set \( S \subset \mathbb{R}^3 \) is a *surface*, if for each \( p_0 = (x_0, y_0, z_0) \in S \), there exists an open neighborhood \( O \) of \( p_0 \) and a \( C^1 \) function \( f : O \to \mathbb{R} \) such that

\[ S \cap O = \{ f(x, y, z) = 0 \} \text{ and } df(p_0) \neq 0, \]
The idea is to approximate \( f \) by its linear approximation at \( p_0 \). For \( (x, y, z) \) close enough to \( (x_0, y_0, z_0) \),

\[
f(x, y, z) \approx f(x_0, y_0, z_0) + (x - x_0)\partial_x f(p_0) + (y - y_0)\partial_y f(p_0) + (z - z_0)\partial_z f(p_0).
\]

3. **Parameterized surface:** We just give the local definition:

**Definition 9.3.** A set \( S \subseteq \mathbb{R}^3 \) is a surface, if for each \( p_0 \in S \), there exists an open domain \( D \subseteq \mathbb{R}^2 \), an open neighborhood \( O \subseteq \mathbb{R}^3 \) of \( p_0 \), and a \( C^1 \) map \( \Phi : D \to \mathbb{R}^3 \) such that

\[
S \cap O = \{ \Phi(u, v) : (u, v) \in D \}
\]

and the Jacobian matrix of \( \Phi \) at \((u_0, v_0)\), where \( \Phi(u_0, v_0) = p_0 \), has maximal rank. In other words,

\[
\partial \Phi(u_0, v_0) = 
\begin{bmatrix}
\partial_u x(u_0, v_0) & \partial_v x(u_0, v_0) \\
\partial_u y(u_0, v_0) & \partial_v y(u_0, v_0) \\
\partial_u z(u_0, v_0) & \partial_v z(u_0, v_0)
\end{bmatrix}
\]

has rank 2.

**9.4 Equivalence of definitions**

If \( S \) is locally a graph, then, near each \( p_0 \in S \),

\[
S = \{ z = h(x, y) \} = \{ f(x, y, z) = 0 \} = \{ (x, y, z) = \Phi(u, v) \},
\]

where

\[
f(x, y, z) = z - h(x, y) \\
\Phi(u, v) = (u, v, h(u, v)).
\]

Therefore, \( S \) is locally both a level set and a parameterized surface.

The converse statements require the implicit and inverse function theorems. We state the exactly versions we need here.

**Theorem 9.4.** *(Implicit function theorem)* Given an open set \( O \subseteq \mathbb{R}^3 \), a point \( p_0 \in O \), and \( C^1 \) function \( f : O \to \mathbb{R} \) such that

\[
f(p_0) = 0 \text{ and } \partial_z f(p_0) \neq 0,
\]

there exists an open neighborhood \( N \subseteq \mathbb{R}^3 \) of \( p_0 \), an open domain \( D \subseteq \mathbb{R}^2 \), and a \( C^1 \) function \( h : D \to \mathbb{R} \) such that, for any \( p \in N \),

\[
\{ f(x, y, z) = 0 : (x, y, z) \in N \} = \{ (x, y, h(x, y)) : (x, y) \in D \}.
\]
Theorem 9.5. (Inverse function theorem) Given an open domain $D \subset \mathbb{R}^2$, a point $(u_0, v_0) \in D$, and a $C^1$ map $\Phi : D \to \mathbb{R}^3$, let
\[
\hat{\Phi} : D \to \mathbb{R}^2 \\
(u, v) \mapsto (x(u, v), y(u, v)).
\]
If the Jacobian matrix $\partial \hat{\Phi}(u_0, v_0)$ has maximal rank (i.e., is invertible), then there exists an open neighborhood $N \subset D$ of $(u_0, v_0)$ and a $C^1$ map $\Psi : \hat{\Phi}(N) \to \mathbb{R}^2$, such that, for any $(x, y) \in \hat{\Phi}(N)$,
\[
\hat{\Phi}(\Psi(x, y)) = (x, y).
\]

9.5 Coordinate charts

A $C^1$ coordinate chart of a surface $S \subset \mathbb{R}^3$ consists of an open domain $D \subset \mathbb{R}^2$, a $C^1$ map.

9.6 Tangent space at a point on a surface

The tangent space at a point $p \in S$ will be denoted $T_pS$. It consists of all possible velocities at $p$ of curves that lie in the surface and pass through $p$. In other words,
\[
T_pS = \{ v \in \mathbb{R}^3 : \text{there exists a } C^1 \text{ curve } c : I \to S \text{ such that } c(0) = p \text{ and } \dot{c}(0) = v \}.
\]

Suppose $D$ is an open neighborhood of $(0, 0) \in \mathbb{R}^2$ and $\Phi : D \to S$ is a coordinate chart, where $\Phi(0) = p_0 \in S$. Then, given any $C^1$ curve $\hat{c} : (-T, T) \to D$, where $\hat{c}(0) = (0, 0)$, $c = \Phi \circ \hat{c} : (-T, T) \to S$ is a $C^1$ curve in $S$. Moreover, since
\[
c(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))),
\]
the velocity of $c$ at $p_0$ is
\[
\dot{c}(0) = \left( \frac{\partial x}{\partial u} \dot{u} + \frac{\partial x}{\partial v} \dot{v}, \frac{\partial y}{\partial u} \dot{u} + \frac{\partial y}{\partial v} \dot{v}, \frac{\partial z}{\partial u} \dot{u} + \frac{\partial z}{\partial v} \dot{v} \right)
\]

\[
= \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{v}
\end{bmatrix}
\]

\[=
\partial \Phi(p_0) \langle \dot{u}, \dot{v} \rangle
\]

where $\langle \dot{u}, \dot{v} \rangle = \dot{c}(0)$. Therefore, the Jacobian of $\Phi$ defines at each $p_0 \in S$, a linear map
\[
\partial \Phi(p_0) : \mathbb{R}^2 \to T_{p_0}S \\
\langle \dot{u}, \dot{v} \rangle \mapsto \partial \Phi(p_0) \langle \dot{u}, \dot{v} \rangle
\]
Since $\partial \Phi(p_0)$ has maximal rank, this map is injective.

Conversely, since $\Phi$ is an injective map, given any $C^1$ curve $c : (\mathcal{T}, T) \to S$, there is a corresponding curve $\hat{c} = \Phi^{-1} \circ c : (-T, T) \to D$. One can use the inverse function theorem to prove that $\hat{c}$ is $C^1$. The argument above shows that $\hat{c}(0) \in T_p S$ lies in the image of the linear map $\partial \Phi(p_0)$. This implies that the map (12.1) is a linear isomorphism.

The tangent space $T_p$ can also described using the definition of the surface $S$ as a level set. Suppose

$$S = \{ p \in O : f(p) = 0 \},$$

where $O$ is an open neighborhood of $p_0$ in $\mathbb{R}^3$ and $f$ is a $C^1$ function on $O$ such that $df \neq 0$. Given any $C^1$ curve $c : (-T, T) \to S \cap O$ such that $c(0) = p_0$ and $\dot{c}(0) = \langle \dot{x}, \dot{y}, \dot{z} \rangle$,

$$0 = \frac{d}{dt} f(c(t)) = \frac{d}{dt} (f(x(t), y(t), z(t))) = \dot{x} \partial_x f + \dot{y} \partial_y f + \dot{z} \partial_z f = \nabla f \cdot \dot{c}. \quad (9.2)$$

It follows that

$$T_p S \subset \{ v \in \mathbb{R}^3 : v \cdot \nabla f(p) = 0 \}. $$

However, since both sides are 2-dimensional linear subspaces, they must be the same. Therefore, they are in fact equal.
Chapter 10
Calculus on Affine Space

10.1 Topology

Given an affine space $\mathbb{A}^m$, there is an affine isomorphism $\Phi : \mathbb{R}^m \to \mathbb{A}^m$. The topology on $\mathbb{A}^m$ is assumed to be the one on $\mathbb{R}^m$ pushed forward to $\mathbb{A}^m$. Specifics follow.

Recall first the following: Given $p_0 \in \mathbb{R}^m$ and $r > 0$, define the open ball of radius $r$ centered at $p_0$ to be $B(p_0, r) = \{p \in \mathbb{R}^m : |p - p_0| < r\}$.

A set $O \subset \mathbb{R}^m$ is open, if and only if, for each $p_0 \in O$, there exists $r > 0$ such that $B(p_0, r) \subset O$.

10.2 Functions and maps

Given an open set $O \subset \mathbb{A}^m$, a function $f : O \to \mathbb{R}$ is $C^k$, where $k \geq 0$, if the function $f \circ \Phi : \Phi^{-1}(O) \to \mathbb{R}$ is $C^k$.

More generally, given affine spaces $\mathbb{A}^m$, $\mathbb{B}^n$ and an open set $O \subset \mathbb{A}^m$, a map $F : O \to \mathbb{B}^n$ is $C^k$, if $\Psi^{-1} \circ F \circ \Phi : \Phi^{-1}(O) \to \mathbb{R}^n$ is $C^k$, where $\Psi : \mathbb{R}^n \to \mathbb{B}^n$ is an affine isomorphism.

10.3 Dual vector space

Definition 10.1. Given a vector space $\mathbb{V}$, its dual vector space $\mathbb{V}^*$ is the set of all linear functions of $\mathbb{V}$,

$$\mathbb{V}^* = \{\theta : \mathbb{V} \to \mathbb{R} : \theta \text{ is linear}\}.$$ 

For convenience, we shall use the following angle bracket notation: Given $\theta \in \mathbb{V}^*$ and $v \in \mathbb{V}$,

$$\langle \theta, v \rangle = \langle v, \theta \rangle = \theta(v).$$

Lemma 10.2. Given any linear map $L : \mathbb{V} \to \mathbb{W}$, there is a unique linear map $L^* : \mathbb{W}^* \to \mathbb{V}^*$, such that, for any $v \in \mathbb{V}$ and $\eta \in \mathbb{W}^*$,

$$\langle \eta, L(v) \rangle = \langle L^*(\eta), v \rangle.$$
Lemma 10.3. There is a natural isomorphism $(\mathbb{V}^*)^* = \mathbb{V}$.

Given a basis $(e_1, \ldots, e_m)$ of $\mathbb{V}$, there is a natural basis $(\theta^1, \ldots, \theta^m)$ of $\mathbb{V}^*$, where, for each $1 \leq i, j \leq m$,

$$\langle \theta^i, e_j \rangle = \delta^i_j.$$  

In particular, if $v = a^i e_i$, then, for each $1 \leq j \leq m$,

$$\langle \theta^j, v \rangle = a^j.$$

### 10.4 Derivatives

Let $O \subset \mathbb{A}^m$ be open.

#### 10.4.1 Directional derivative

Given a $C^1$ function $f : O \to \mathbb{R}$ and a tangent vector $v \in \dot{\mathbb{A}}^m$, its directional derivative at $p \in O$ with velocity $v$ is defined to be

$$d_v f(p) = \left. \frac{df}{dt} \right|_{t=0} f(c(t)) = \lim_{t \to 0} \frac{f(c(t)) - f(c(0))}{t},$$  

(10.1)

where $c : (-T, T) \to \mathbb{A}^m$ is a $C^1$ curve such that $c(0) = p$ and $\dot{c}(0) = v$.

**Lemma 10.4.** If $c_1$ and $c_2$ are curves that satisfy the assumptions above, then they give the same value for $d_v f(p)$ in (10.1).

#### 10.4.2 Differential of a function

Given a $C^1$ function $f : O \to \mathbb{R}$ and $p \in O$, the function

$$df(p) : \dot{\mathbb{A}} \to \mathbb{R}$$

$$v \mapsto d_v f(p)$$

is a linear function of $v \in \dot{\mathbb{A}}$. Therefore, $df(p) \in \dot{\mathbb{A}}^*$. We can therefore, define the differential of a $C^1$ function $f : O \to \mathbb{R}$ to be the map

$$df : O \to \dot{\mathbb{A}}$$

$$p \mapsto df(p),$$

where, for each $v \in \dot{\mathbb{A}},$

$$\langle df(p), v \rangle = d_v f(p),$$

as defined above.
10.5 Differential 1-form on an open subset of $\mathbb{A}^m$

**Definition 10.5.** Given an open set $O \subset \mathbb{A}^m$, a differential 1-form is a map $\omega : O \to \mathbb{V}^*$. 

**Example.** The differential of a function is a 1-form.

10.6 Line integral of a 1-form along a curve in $\mathbb{A}^m$

**Definition 10.6.** If $O \subset \mathbb{A}^m$ is open, $c : [0, T] \to O$ is a $C^1$ curve, and $\omega : O \to (\mathbb{V}^m)^*$ a 1-form, then the line integral of $\omega$ along $c$ is defined to be

$$\int_c \omega = \int_0^T \langle \omega(c(t)), c'(t) \rangle \, dt.$$ 

Observe that the integrand on the right is a real-valued function of $t \in [0, T]$ and therefore the integral is standard single variable integral.

By using the same proof of the chain rule for functions of a single variable, the following holds:

**Lemma 10.7.** If $c : I \to \mathbb{A}^m$ is a $C^1$ curve and $\phi : I' \to I$ is a $C^1$ function on the interval $I'$, then

$$(c \circ \phi)'(t) = c'(\phi(t))\phi'(t).$$

The same proof for the change of variables for a single variable integral now implies:

**Theorem 10.8.** If $O \subset \mathbb{A}^m$ is open, $c : [0, T] \to O$ is a $C^1$ curve, and $\omega : O \to (\mathbb{V}^m)^*$ a 1-form, then for any $C^1$ function $\phi : [0, \hat{T}] \to [0, T]$ such that $\phi(0) = 0$ and $\phi(\hat{T}) = T$,

$$\int_{\hat{c}} \omega = \int_c \omega,$$

where $\hat{c} = c \circ \phi$.

The fundamental theorem of calculus implies the following:

**Theorem 10.9.** If $c : [0, T] \to \mathbb{A}^m$ is a $C^1$ curve and $f : O \to \mathbb{R}$ a $C^1$ function on an open set $O \subset \mathbb{A}^m$, then

$$\int_c df = f(c(T)) - f(c(0)).$$

In particular, the line integral of $df$ from $p$ to $q$ does not depend on which $C^1$ curve is used to connect $p$ to $q$. 
10.7 Fundamental example

Recall that, associated with an affine basis \( (p_0, \ldots, p_m) \) of \( \mathbb{A}^m \) is a basis \( E = (e_1 - 1, \ldots, e_m) \) of \( \mathbb{V}^m \), where

\[
e_1 = p_1 - p_0, \ldots, e_m = p_m - p_0.
\]

On the other hand, recall that \( I^{-1}_E(p) = (x^1, \ldots, x^m) \), where for each \( 1 \leq i \leq m \), \( t \)

\[
x^i : \mathbb{A}^m \rightarrow \mathbb{R}
\]

\[
p_0 + v \mapsto v^i.
\]

The differential of \( x^i \) is defined to be \( dx^i \in (\mathbb{V}^m)^* \), where

\[
\langle dx^i(p), v \rangle = \frac{d}{dt} \Big|_{t=0} x^i(p + tv) = v^i.
\]

Equivalently,

\[
\langle dx^i, e_j \rangle = \delta^i_j.
\]

and, therefore, \( (dx^1, \ldots, dx^m) \) is the basis of \( (\mathbb{V}^m)^* \) dual to the basis \( E = (e_1, \ldots, e_m) \) of \( \mathbb{V}^m \).

Given a \( C^1 \) function \( f : O \rightarrow \mathbb{R} \), note that, if \( p_0 \in O \) and \( I_E(x^1, \ldots, x^n) = p \), then

\[
\partial_i((f \circ I_E)(x^1, \ldots, x^n)) = \frac{d}{dt} \bigg|_{t=0} f(p + te_i)
\]

\[
= \langle df(p), e_i \rangle.
\]

Therefore,

\[
df(p) = \langle df(p), e_i \rangle dx^i.
\]

This leads to the following notation: Given an affine basis \( P = (p_0, \ldots, p_m) \) of \( \mathbb{A}^m \), instead of denoting the corresponding basis of \( \mathbb{V}^m \) by \( E = (e_1, \ldots, e_m) \), we denote it by \( \partial = (\partial_1, \ldots, \partial_m) \), where, for each \( i = 1, \ldots, m \),

\[
\partial_i = p_i - p_0.
\]

As noted above, its dual basis is \( dx = (dx^1, \ldots, dx^m) \). Given a \( C^1 \) function \( f : O \rightarrow \mathbb{R} \), we denote

\[
\partial_i f(p) = \langle df(p), \partial_i \rangle.
\]

Therefore, omitting the \( p \) for brevity,

\[
df = \partial_i f dx^i.
\]
Chapter 11

Differential of a map

11.1 Characterization of a linear map

Lemma 11.1. Given a map \( F : \mathbb{V}^m \rightarrow \mathbb{W}^n \), the following are equivalent:

- \( F \) is linear.
- \( \ell \circ F : \mathbb{V}^m \rightarrow \mathbb{R} \) is linear for any \( \ell \in (\mathbb{W}^n)^* \).
- Given a basis \( \ell^1, \ldots, \ell^n \) of \( \mathbb{W}^n \), \( \ell_i \circ F \) is linear for each \( i = 1, \ldots, n \).

11.1.1 Differential of a map

Given affine spaces \( \mathbb{A}, \mathbb{B} \) and an open set \( O \subset \mathbb{A} \), let \( F : O \rightarrow \mathbb{B} \) be a \( C^1 \) map. Given \( p \in O \), we can define a linear map

\[ \partial_p F : \dot{\mathbb{A}} \rightarrow \dot{\mathbb{B}} \]

as follows: Given \( v \in \dot{\mathbb{A}} \), let \( c : (-T, T) \rightarrow \mathbb{A} \) be a \( C^1 \) curve such that \( c(0) = p \) and \( \dot{c}(0) = v \). Then \( F \circ c : (-T, T) \rightarrow \mathbb{B} \) is also a \( C^1 \) curve. We then define \( \partial_p F(v) \) to be the velocity of \( F \circ c \) at \( t = 0 \),

\[ \partial_p F(v) = \frac{d}{dt} \bigg|_{t=0} F(c(t)). \]

This linear map is also known as the Jacobian of \( F \). If \( \mathbb{A} = \tilde{\mathbb{R}}^m \) and \( \mathbb{B} = \tilde{\mathbb{R}}^n \), then \( \partial F_p \) is the \( n \)-by-\( m \) matrix of partial derivatives of \( F(x^1, \ldots, x^m) = (F^1(x^1, \ldots, x^m), \ldots, F^n(x^1, \ldots, x^m)) \).

11.1.2 Pushforward of a tangent vector

Given \( p \in O \) and \( v \in \dot{\mathbb{A}} \), we will also write

\[ F_* v = \partial_p F(v) \]

and call it the *pushforward* of \( v \) by \( F \).
11.1.3 Pullback of a cotangent vector

Given $p \in O$ and $\theta \in \mathbb{A}^*$, we will write

$$F^*\theta = (\partial_p F)^t(\theta).$$

11.2 Chain Rule

Lemma 11.2. If $F : \mathbb{A}^m \to \mathbb{B}^n$ and $G : \mathbb{B}^n \to \mathbb{C}$ are $C^1$ maps, then so is $G \circ F$, and

$$d(G \circ F)(p) = dG(F(p)) \circ dF(p).$$
Chapter 12

Surfaces in affine 3-space

12.1 Preliminaries

A set $O \subseteq \mathbb{R}^m$ is called open, if, for any $p \in O$, there is an $r > 0$ such that

$$B(p, r) \subseteq O.$$ 

Think of an open set as one that does not contain any point on its boundary. A set $D \subseteq \mathbb{R}^m$ is called an open domain, if it is an open set. Given a point $p \in \mathbb{R}^m$, an open neighborhood of $p$ is an open set that contains $p$.

12.2 Surfaces in Cartesian 3-space

12.2.1 Informal but flawed descriptions of a surface

1. **Graph:** Given an open domain $D \subseteq \mathbb{R}^2$ and a $C^1$ function $h : D \to \mathbb{R}$, let

$$S = \{z = h(x, y)\} = \{(x, y, h(x, y)) : (x, y) \in D\}.$$ 

**Example:**

$$S = \left\{ z = \frac{1}{\sqrt{1 - x^2 - y^2}} \right\}.$$ 

2. **Level set:** Given an open set $O \subseteq \mathbb{R}^3$ and a $C^1$ function $f : O \to \mathbb{R}$, let

$$S = \{ f(x, y, z) = 0 \} = \{(x, y, z) \in O : f(x, y, z) = 0\}.$$ 

**Examples:**

$$S = \{ x^2 + y^2 + z^2 - 1 = 0 \}$$

$$S = \{ x^2 + y^2 - z^2 = 0 \}.$$
3. **Parameterized surface:** Given an open domain $D \subset \mathbb{R}^2$ and a $C^1$ map $\Phi : D \rightarrow \mathbb{R}^3$,

$$S = \Phi(D) = \{(\Phi(u, v) : (u, v) \in D) = \{(x(u, v), y(u, v), z(u, v) : (u, v) \in D\},$$

where

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

**Example:**

$$S = \{(r \cos \theta, r \sin \theta, r)\}.$$  

But these descriptions are *FLAWED*. And are they equivalent? How can these descriptions be made abstract for a surface in $\mathbb{A}^3$?

### 12.2.2 2-planes in 3-space

The simplest surface in $\mathbb{R}^3$ is a 2-plane. It can be described as follows:

1. **Graph:** Given an affine function

   $$h(x, y) = ax + by + c,$$

   let

   $$S = \{z = h(x, y)\} = \{z = ax + by + c\}.$$

   **Examples:**

   $$S = \{z = 0\}$$
   $$S = \{z = x + y + 1\}.$$

   The only shortcoming is that vertical planes are not graphs. This can be resolved by allowing $x$ or $y$ be the output variable and the other two variables be the input variables.

2. **Level set.** Given an affine function

   $$f(x, y, z) = \alpha x + \beta y + \delta z + \gamma,$$

   let

   $$S = \{f(x, y, z) = 0\} = \{\alpha x + \beta y + \delta z + \gamma = 0\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} \alpha & \beta & \delta \end{bmatrix} + \gamma = 0 \right\}.$$

   This, however, is not always a surface, because, if $\alpha = \beta = \delta = 0$, then $S$ is not a 2-plane. An extra assumption is needed, namely $\langle \alpha, \beta, \gamma \rangle \neq 0$. We can write this as follows: First, recall that the differential of $f$ is defined to be

   $$df = \partial_x f \, dx + \partial_y f \, dy + \partial_z f \, dz.$$

   In order for $S$ to be a plane and therefore a surface, we need to assume that

   $$df \neq 0.$$
3. **Parameterized surface.** Given an affine map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\Phi(u, v) = (x_0, y_0, z_0) + A(u, v),$$

where $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $A = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \\ a_3^1 & a_3^2 \end{bmatrix}$, let

$$S = \{ \Phi(u, v) : \langle u, v \rangle \} = \{(x_0, y_0, z_0) + A(u, v), \langle u, v \rangle \in \mathbb{R}^2 \} = \{(x_0 + a_1^1 u + a_1^2 v, y_0 + a_2^1 u + a_2^2 v, z_0 + a_3^1 u + a_3^2 v)\}.$$

This, however, does not always work. If the rank of $A$ is not maximal, i.e., 2, then $S$ is either a point or a line. We must therefore assume that the rank of $A$ is maximal. Equivalently, we must assume $\ker A = \{0\}$.

The assumptions needed for the function $f$ and map $\Phi$ are nondegeneracy conditions. Notice that they can be expressed in terms of the differentials of $f$ and $\Phi$. In particular, the set $S = \{f = 0\}$ is a surface, if

$$df \neq 0.$$

The set $S = \Phi(\mathbb{R}^2)$ is a surface, if the Jacobian of $\Phi$,

$$\partial \Phi = \begin{bmatrix} \partial_{u}x & \partial_{v}x \\ \partial_{u}y & \partial_{v}y \\ \partial_{u}z & \partial_{v}z \end{bmatrix}$$

has maximal rank.

### 12.2.3 Local descriptions of a surface

1. **Graph:** Given an open domain $D \subset \mathbb{R}^2$ and $C^1$ function $h : D \to \mathbb{R}$, the set

$$S = \{z = h(x, y)\}$$

is a surface. It suffices to do this locally.

**Definition 12.1.** A set $S \subset \mathbb{R}^3$ is a surface, if for any $p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, there exists an open neighborhood $O \subset \mathbb{R}^3$ of $p_0$, an open domain $D \subset \mathbb{R}^2$, and a $C^1$ function $f : D \to \mathbb{R}$ such that

$$S \cap O = \{z = f(x, y)\}.$$

2. **Level set:** We start with a global definition. If

$$S = \{f(x, y, z) = 0\},$$

where $O \subset \mathbb{R}^3$ is open and $f : O \to \mathbb{R}$ is $C^1$, then $S$ is a surface.

The equivalent local definition is the following:
Definition 12.2. A set $S \subset \mathbb{R}^3$ is a surface, if for each $p_0 = (x_0, y_0, z_0) \in S$, there exists an open neighborhood $O$ of $p_0$ and a $C^1$ function $f : O \to \mathbb{R}$ such that

$$S \cap O = \{f(x, y, z) = 0\} \text{ and } df(p_0) \neq 0,$$

The idea is to approximate $f$ by its linear approximation at $p_0$. For $(x, y, z)$ close enough to $(x_0, y_0, z_0)$,

$$f(x, y, z) \approx f(x_0, y_0, z_0) + (x - x_0)\partial_x f(p_0) + (y - y_0)\partial_y f(p_0) + (z - z_0)\partial_z f(p_0).$$

3. Parameterized surface: We just give the local definition:

Definition 12.3. A set $S \subset \mathbb{R}^3$ is a surface, if for each $p_0 \in S$, there exists an open domain $D \subset \mathbb{R}^2$, an open neighborhood $O \subset \mathbb{R}^3$ of $p_0$, and a $C^1$ map $\Phi : D \to \mathbb{R}^3$ such that

$$S \cap O = \{\Phi(u, v) : (u, v) \in D\}$$

and the Jacobian matrix of $\Phi$ at $(u_0, v_0)$, where $\Phi(u_0, v_0) = p_0$, has maximal rank. In other words,

$$\partial \Phi(u_0, v_0) = \begin{bmatrix}
\partial_u x(u_0, v_0) & \partial_v x(u_0, v_0) \\
\partial_u y(u_0, v_0) & \partial_v y(u_0, v_0) \\
\partial_u z(u_0, v_0) & \partial_v z(u_0, v_0)
\end{bmatrix}$$

has rank 2.

12.2.4 Equivalence of definitions

If $S$ is locally a graph, then, near each $p_0 \in S$,

$$S = \{z = h(x, y)\} = \{f(x, y, z) = 0\} = \{(x, y, z) = \Phi(u, v)\},$$

where

$$f(x, y, z) = z - h(x, y)$$

$$\Phi(u, v) = (u, v, h(u, v)).$$

Therefore, $S$ is locally both a level set and a parameterized surface.

The converse statements require the implicit and inverse function theorems. We state the exactly versions we need here.

Theorem 12.4. (Implicit function theorem) Given an open set $O \subset \mathbb{R}^3$, a point $p_0 \in O$, and $C^1$ function $f : O \to \mathbb{R}$ such that

$$f(p_0) = 0 \text{ and } \partial_z f(p_0) \neq 0,$$

there exists an open neighborhood $N \subset \mathbb{R}^3$ of $p_0$, an open domain $D \subset \mathbb{R}^2$, and a $C^1$ function $h : D \to \mathbb{R}$ such that, for any $p \in N$,

$$\{f(x, y, z) = 0 : (x, y, z) \in N\} = \{(x, y, h(x, y)) : (x, y) \in D\}.$$
Theorem 12.5. (Inverse function theorem) Given an open domain \( D \subseteq \mathbb{R}^2 \), a point \((u_0, v_0) \in D\), and a \( C^1 \) map \( \Phi : D \to \mathbb{R}^3 \), let
\[
\hat{\Phi} : D \to \mathbb{R}^2 \\
(u, v) \mapsto (x(u, v), y(u, v)).
\]

If the Jacobian matrix \( \partial \hat{\Phi}(u_0, v_0) \) has maximal rank (i.e., is invertible), then there exists an open neighborhood \( N \subseteq D \) of \((u_0, v_0)\) and a \( C^1 \) map \( \Psi : \hat{\Phi}(N) \to \mathbb{R}^2 \), such that, for any \((x, y) \in \hat{\Phi}(N)\),
\[
\Phi(\Psi(x, y)) = (x, y).
\]

12.2.5 Coordinate charts

A \( C^1 \) coordinate chart of a surface \( S \subseteq \mathbb{R}^3 \) consists of an open domain \( D \subseteq \mathbb{R}^2 \), a \( C^1 \) map.

12.2.6 Tangent space at a point on a surface

The tangent space at a point \( p \in S \) will be denoted \( T_pS \). It consists of all possible velocities at \( p \) of curves that lie in the surface and pass through \( p \). In other words,
\[
T_pS = \{v \in \mathbb{R}^3 : \text{there exists a } C^1 \text{ curve } c : I \to S \text{ such that } c(0) = p \text{ and } \dot{c}(0) = v\}.
\]

Suppose \( D \) is an open neighborhood of \((0, 0) \in \mathbb{R}^2 \) and \( \Phi : D \to S \) is a coordinate chart, where \( \Phi(0) = p_0 \in S \). Then, given any \( C^1 \) curve \( \hat{c} : (-T, T) \to D \), where \( \hat{c}(0) = (0, 0) \), \( c = \Phi \circ \hat{c} : (-T, T) \to S \) is a \( C^1 \) curve in \( S \). Moreover, since
\[
c(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))),
\]
The velocity of \( c \) at \( p_0 \) is
\[
\dot{c}(0) = \left\langle \frac{\partial x}{\partial u} \dot{u} + \frac{\partial x}{\partial v} \dot{v}, \frac{\partial y}{\partial u} \dot{u} + \frac{\partial y}{\partial v} \dot{v}, \frac{\partial z}{\partial u} \dot{u} + \frac{\partial z}{\partial v} \dot{v} \right\rangle
\]
\[
= \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{v}
\end{bmatrix}
= \partial \Phi(p_0) \langle \dot{u}, \dot{v} \rangle
\]
where \( \langle \dot{u}, \dot{v} \rangle = \dot{c}'(0) \). Therefore, the Jacobian of \( \Phi \) defines at each \( p_0 \in S \), a linear map
\[
\partial \Phi(p_0) : \mathbb{R}^2 \to T_{p_0}S \\
\langle \dot{u}, \dot{v} \rangle \mapsto \partial \Phi(p_0) \langle \dot{u}, \dot{v} \rangle
\] (12.1)

Since \( \partial \Phi(p_0) \) has maximal rank, this map is injective.
Conversely, since \( \Phi \) is an injective map, given any \( C^1 \) curve \( c : (-T, T) \to S \), there is a corresponding curve \( \hat{c} = \Phi^{-1} \circ c : (-T, T) \to D \). One can use the inverse function theorem to prove that \( \hat{c} \) is \( C^1 \). The argument above shows that \( \hat{c}(0) \in T_p S \) lies in the image of the linear map \( \partial \Phi(p_0) \). This implies that the map \( (12.1) \) is a linear isomorphism.

The tangent space \( T_p \) can also described using the definition of the surface \( S \) as a level set. Suppose

\[
S = \{ p \in O : f(p) = 0 \},
\]

where \( O \) is an open neighborhood of \( p_0 \) in \( \mathbb{R}^3 \) and \( f \) is a \( C^1 \) function on \( O \) such that \( df \neq 0 \). Given any \( C^1 \) curve \( c : (-T, T) \to S \cap O \) such that \( c(0) = p_0 \) and \( \dot{c}(0) = (\dot{x}, \dot{y}, \dot{z}) \),

\[
0 = \frac{d}{dt} f(c(t)) = \frac{d}{dt} (f(x(t), y(t), z(t))) = \dot{x} \partial_x f + \dot{y} \partial_y f + \dot{z} \partial_z f = \nabla f \cdot \dot{c}.
\] (12.2)

It follows that

\[
T_p S \subset \{ v \in \mathbb{R}^3 : v \cdot \nabla f(p) = 0 \}.
\]

However, since both sides are 2-dimensional linear subspaces, they must be the same. Therefore, they are in fact equal.
Chapter 13

Surfaces in Euclidean 3-space

In the previous chapter, we never used any geometric concepts such as distance, length, or angle. In this chapter, we use them to study the geometry of a surface in Euclidean 3-space. Specifically, we use here the standard dot product on $\mathbb{R}^3$.

13.1 First fundamental form

The first fundamental form is simply the dot product restricted to the subspace $T_pS \subset \mathbb{R}^3$. We want, however, to understand what it looks like using a coordinate chart $\Phi : D \to \mathbb{R}^3$. In particular, we want to “pull back” the dot product to the vector space $\mathbb{R}^2$, viewed as velocities of curves in $D$.

First, recall that the dot product of any two vectors $\langle \dot{x}_1, \dot{y}_1, \dot{z}_1 \rangle$ and $\langle \dot{x}_2, \dot{y}_2, \dot{z}_2 \rangle$ is

$$\langle \dot{x}_1, \dot{y}_1, \dot{z}_1 \rangle \cdot \langle \dot{x}_2, \dot{y}_2, \dot{z}_2 \rangle = \dot{x}_1 \dot{x}_2 + \dot{y}_1 \dot{y}_2 + \dot{z}_1 \dot{z}_2.$$

Therefore, given any two $\hat{c}_1, \hat{c}_2 : (-T,T) \to D$ such that $\hat{c}_1(0) = \hat{c}_2(0) = p_0 \in S$, the dot
product of the velocities $\dot{c}_1 = (\Phi \circ \dot{c}_1)'(0)$ and $\dot{c}_2 = (\Phi \circ \dot{c}_2)'(0)$ at $p_0$ is given by

$$
\dot{c}_1(0) \cdot \dot{c}_2(0) = (\Phi \circ \dot{c}_1)'(0) \cdot (\Phi \circ \dot{c}_2)'(0)
= (\partial \Phi_1'(0)) \cdot (\partial \Phi_2'(0))
= \left( \frac{\partial x}{\partial u} \dot{u}_1 + \frac{\partial x}{\partial v} \dot{v}_1 \right) \left( \frac{\partial x}{\partial u} \dot{u}_2 + \frac{\partial x}{\partial v} \dot{v}_2 \right) + \left( \frac{\partial y}{\partial u} \dot{u}_1 + \frac{\partial y}{\partial v} \dot{v}_1 \right) \left( \frac{\partial y}{\partial u} \dot{u}_2 + \frac{\partial y}{\partial v} \dot{v}_2 \right)
+ \left( \frac{\partial z}{\partial u} \dot{u}_1 + \frac{\partial z}{\partial v} \dot{v}_1 \right) \left( \frac{\partial z}{\partial u} \dot{u}_2 + \frac{\partial z}{\partial v} \dot{v}_2 \right)
= \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} \right) \dot{u}_1 \dot{u}_2 + \dot{v}_1 \dot{v}_2
+ \left( \frac{\partial x}{\partial v} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial v} \frac{\partial z}{\partial v} \right) (\dot{v})^2
= \left[ \begin{array}{l} \dot{u}_1 \\ \dot{v}_1 \end{array} \right]
\left[ \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{array} \right]
\left[ \begin{array}{c} \dot{u}_2 \\ \dot{v}_2 \end{array} \right]
= \langle \dot{u}_1, \dot{v}_1 \rangle \cdot (\partial \Phi)^T (\partial \Phi)(\dot{u}_2, \dot{v}_2),
$$

where $(\partial \Phi)^T$ denotes the transpose of the matrix $\partial \Phi$.

Therefore, if, for each $p \in S$, we define the symmetric 2-by-2 matrix

$$
G(p) = (\partial \Phi(p))^T (\partial \Phi(p)),
$$

and, given any tangent vectors $\langle \dot{u}_1, \dot{v}_1 \rangle, \langle \dot{u}_2, \dot{v}_2 \rangle \in T_pS$, define

$$
g(p)(\langle \dot{u}_1, \dot{v}_1 \rangle, \langle \dot{u}_2, \dot{v}_2 \rangle) = \langle \dot{u}_1, \dot{v}_1 \rangle \cdot G(\dot{u}_2, \dot{v}_2),
$$

then,

$$
\langle ((\partial \Phi(p))\dot{u}_1, \dot{v}_1) \cdot ((\partial \Phi(p))\dot{u}_2, \dot{v}_2) = g(p)(\langle \dot{u}_1, \dot{v}_1 \rangle, \langle \dot{u}_2, \dot{v}_2 \rangle).
$$

### 13.2 Change of notation

The calculation above is a mess, using several different variables. We introduce new notation that is easier to write and has the advantage that it makes generalization to higher dimensions easy.

From now on, a point in $\mathbb{R}^2$ will be denoted $x = (x^1, x^2)$, and a vector in $\mathbb{R}^2$ by $\dot{x} = (\dot{x}^1, \dot{x}^2)$. A point in $\mathbb{R}^3$ will be denoted $y = (y^1, y^2, y^3)$, and a vector in $\mathbb{R}^3$ by $\dot{y} = (\dot{y}^1, \dot{y}^2, \dot{y}^3)$. A coordinate chart will be denoted $y(x) = (y^1(x^1, x^2), y^2(x^1, x^2), y^3(x^1, x^2))$.

If we fix $x^2$, then the map $t \mapsto y(t, x^2)$ defines a curve in $S$. This defines a family of curves on the coordinate chart. There is a second family of curves defined by fixing $x^1$ and
using \( x^2 \) as the parameter of a curve. The velocities of these curves are the partial derivatives of \( y \),
\[
\frac{\partial y}{\partial x_1} \text{ and } \frac{\partial y}{\partial x_2},
\]
which are the columns of the Jacobian
\[
\frac{\partial y}{\partial x} = \begin{bmatrix}
\frac{\partial y^1}{\partial x_1} & \frac{\partial y^1}{\partial x_2} \\
\frac{\partial y^2}{\partial x_1} & \frac{\partial y^2}{\partial x_2} \\
\frac{\partial y^3}{\partial x_1} & \frac{\partial y^3}{\partial x_2}
\end{bmatrix}.
\]
That the Jacobian has rank 2 implies that
\[
\begin{pmatrix}
\frac{\partial y}{\partial x_1} \\
\frac{\partial y}{\partial x_2}
\end{pmatrix}
\]
is, for each \( p \in S \), a basis of \( T_pS \). Moreover, the dot product of
\[
v_1 = v_1^1 \frac{\partial y}{\partial x_1} + v_1^2 \frac{\partial y}{\partial x_2}, \quad v_2 = v_2^1 \frac{\partial y}{\partial x_1} + v_2^2 \frac{\partial y}{\partial x_2},
\]
is given by
\[
v_1 \cdot v_2 = \left( v_1^1 \frac{\partial y}{\partial x_1} + v_1^2 \frac{\partial y}{\partial x_2} \right) \cdot \left( v_2^1 \frac{\partial y}{\partial x_1} + v_2^2 \frac{\partial y}{\partial x_2} \right) = g_{ij}(p)v_i^iv_j^j,
\]
where, for each \( 1 \leq i, j \leq 2 \),
\[
g_{ij} = \frac{\partial y}{\partial x_i} \cdot \frac{\partial y}{\partial x_j}.
\] (13.1)

It follows that the length of any tangent vector or curve at \( p \in S \) and the angle of any pair of vectors in \( T_pS \) can be determined using the coordinates \((x^1, x^2)\) and the symmetric positive definite matrix \( G = [g_{ij}(p)] \).

More generally, an inner product on \( T_pS \), for each \( p \in S \), is called a Riemannian metric. Given a Riemannian metric \( g, p \in S \), and a basis \( e_1, \ldots, e_m \in T_pS \), there is a positive definite matrix \( G(p) = [g_{ij}(p)] \), where
\[
g_{ij}(p) = e_i \cdot e_j.
\]
Such a Riemannian metric need not be the same as the one given by (13.1). Moreover, given a Riemannian metric \( g \), there can be more than one embedding that satisfies (13.1). For these reasons, we call a Riemannian metric an intrinsic geometric structure on \( S \).
13.3 Gauss map

If a surface $S \subset \mathbb{R}^3$ is defined as the level set of a $C^2$ function $f$, then the Gauss map is defined to be

$$
\eta : S \to \mathbb{R}^3 \\
p \mapsto \frac{\nabla f(p)}{|\nabla f(p)|},
$$

For each $p \in S$, the vector $\eta(p)$ is a unit vector orthogonal to $T_pS$. It is unique up to sign.

13.4 Geometry of a curve on a surface

Recall that the curvature of a curve in $\mathbb{E}^2$ and the curvature and torsion of a unit speed curve in $\mathbb{E}^3$ uniquely determine the shape of the curve. Moreover, the definition of these functions does not depend on how the curve is described (i.e., the parameterization of the curve or the coordinates in Euclidean space). The analogous geometric invariants for a surface in $\mathbb{E}^3$ are called the first and second fundamental forms. The first fundamental form plays a role similar to the unit speed parameterization of a curve. The second fundamental form is analogous to the curvature and torsion of a curve. We will see later that the first and second fundamental forms uniquely determine the shape of the surface.

Before describing the second fundamental form of a surface $S$, we note that the first fundamental form at each point $p \in S$ is simply the dot product on $\mathbb{E}^3$ but restricted to vectors tangent to $S$ at $p$. Its

Given a $C^2$ surface $S \subset \mathbb{R}^3$, a point $p \in S$, and a unit tangent vector $u \in T_pS$, consider a $C^2$ unit speed curve $c : I \to S$ such that $c(0) = p$ and $\dot{c}(0) = u$. We can define an orthonormal frame $(e_1, e_2, e_3)$ along $c$, where

$$
e_1(t) = \dot{c}(t) \\
e_3(t) = \eta(c(t)).
$$

This is not the Frenet-Serret frame, and therefore we have to recompute the derivatives of $e_1, e_2, e_3$ with respect to $t$.

First, since $e_1$ is unit, it follows by Lemma 7.4 that $\dot{e}_1$ is orthogonal to $e_1$ and, therefore, there are functions $\kappa_2$ and $\kappa_3$ such that

$$
\ddot{c}(t) = \dot{e}_1(t) = \kappa_2(t)e_2(t) - \kappa_3(t)e_3(t).
$$

Since $e_2$ is tangent to $S$ and $e_3$ is normal to $S$, we call $\kappa_2$ the tangential curvature and $\kappa_3$ the normal curvature to the curve $c$, relative to the surface $S$. The tangential curvature measures how quickly $e_1$ is twisting in the tangent plane, and the normal curvature measures how quickly $e_1$ is twisting out of the tangent plane.
If $S$ is the level set of a $C^2$ function $f$, then, differentiating (12.2), we get

$$0 = \frac{d^2}{dt^2} f(x(t), y(t), z(t))$$

$$= \frac{d}{dt} \left( (\partial_x f, \partial_y f, \partial_z f) \cdot (\dot{x}, \dot{y}, \dot{z}) \right)$$

$$= \frac{\dot{x}}{\dot{t}} \frac{\ddot{x}}{\dot{t}} + \frac{\dot{y}}{\dot{t}} \frac{\ddot{y}}{\dot{t}} + \frac{\dot{z}}{\dot{t}} \frac{\ddot{z}}{\dot{t}} + \langle \partial_x f, \partial_y f, \partial_z f \rangle \cdot \langle \dddot{x}, \dddot{y}, \dddot{z} \rangle$$

$$= \dot{c} \cdot (\partial^2 f) \dot{c} + \nabla f \cdot \dddot{c}$$

$$= \cdot c \cdot (\partial^2 f) \dot{c} - |\nabla f| e_3 \cdot (\kappa_2 e_2 + \kappa_3 e_3)$$

$$= \cdot c \cdot (\partial^2 f) \dot{c} - |\nabla f| e_3$$

where

$$\partial^2 f = \begin{bmatrix}
\partial^2_{xx} f & \partial^2_{xy} f & \partial^2_{xz} f \\
\partial^2_{yx} f & \partial^2_{yy} f & \partial^2_{yz} f \\
\partial^2_{zx} f & \partial^2_{zy} f & \partial^2_{zz} f
\end{bmatrix}$$

is the Hessian of $f$. Therefore, setting $t = 0$, we get

$$\kappa_3(0) = -u \cdot H(p)u,$$

where

$$H = \frac{\partial^2 f}{|\nabla f|}$$

is a 2-by-2 symmetric matrix. This shows that the normal curvature at each point of the curve depends only on the direction of the curve at that point. It does not depend on the shape of the curve at all.

### 13.5 Second fundamental form

We can also show that, for any $v \in T_p S$, $v \cdot Hv$ depends only on the surface $S$ and not on the function $f$ as follows: If $S$ is also the level set of another $C^2$ function $g$, then, by Lemma F.8, there exists a $C^2$ function $\phi$ such that $g = \phi f$. Flipping the sign of $g$, if necessary, we can assume that, along $S$, $\nabla g$ points in the same direction as $\nabla f$. That implies $\phi > 0$. Therefore, along $S$,

$$\frac{\partial^2 g}{|\nabla g|} = \frac{\partial^2 g}{|\nabla g|}$$

$$= \frac{\partial^2 (\phi f)}{|\nabla (\phi f)|}$$

$$= \phi \partial^2 f + \nabla \phi \otimes \nabla f + \nabla f \otimes \nabla \phi + f \partial^2 \phi$$

$$= \phi \partial^2 f + \nabla \phi \otimes \nabla f + \nabla f \otimes \nabla \phi$$

$$= \phi \frac{\partial^2 f + \nabla \phi \otimes \nabla f + \nabla f \otimes \nabla \phi}{\phi |\nabla f|},$$
where we use the following notation: Given functions \( f \) and \( g \),
\[
\nabla f \otimes \nabla g = \begin{bmatrix}
    \partial_x f \partial_x g & \partial_x f \partial_y g & \partial_x f \partial_z g \\
    \partial_y f \partial_x g & \partial_y f \partial_y g & \partial_y f \partial_z g \\
    \partial_z f \partial_x g & \partial_z f \partial_y g & \partial_z f \partial_z g
\end{bmatrix}.
\]

Therefore, if \( p \in S \) and \( v \in T_p S \), then \( f(p) = 0 \) and \( v \cdot \nabla f = 0 \). It follows that
\[
\frac{v \cdot (\partial^2 g)v}{|\nabla g|} = \frac{v \cdot (\phi \partial^2 f + \nabla \phi \otimes \nabla f + \nabla f \otimes \nabla \phi)v}{\phi|\nabla f|} = \frac{v \cdot \partial^2 f v + 2(v \cdot \nabla \phi)(v \cdot \nabla f)}{|\nabla f|}
\]

We can therefore define, for each \( p \), a bilinear function
\[
H(p) : T_p S \times T_p S \to \mathbb{R}
\]
\[
(v, w) \mapsto \frac{v \cdot (\partial^2 f)w}{|\nabla f|}.
\]

This is called the second fundamental form of the surface \( S \) at \( p \in S \). It is a geometric invariant of \( S \) that measures how curved the surface is. In particular, if \( c \) is a unit speed curve on \( S \), then its normal curvature at a point \( c(t) \) is
\[
\kappa_3 = -v \cdot Hv,
\]
where \( v = \dot{c}(t) \). This implies that how quickly the direction of \( c \) twists out of the tangent plane of \( S \) depends only on the geometry of \( S \) and the direction of \( c \).

Given any orthonormal basis of \( T_p S \), \( H(p) \) can be written as a symmetric 2-by-2 matrix, which always has 2 real eigenvalues. These are called the principal curvatures. They represent the maximum and minimum possible values of the normal curvature at a point.

**Example.** The sphere of radius \( r \) centered at the origin is the level set \( f = r^2 \) for the function
\[
f(x, y, z) = x^2 + y^2 + z^2.
\]
The gradient of \( f \) is
\[
\nabla f = 2\langle x, y, z \rangle,
\]
and therefore
\[
|\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2r
\]
\[
\frac{\partial^2 f}{|\nabla f|} = \frac{1}{2r} \begin{bmatrix}
    2 & 0 & 0 \\
    0 & 2 & 0 \\
    0 & 0 & 2
\end{bmatrix}
\]
\[
= \frac{1}{r} I,
\]
where \( I \) is the 3-by-3 identity matrix. Since
\[
v \cdot \left( \frac{1}{r} I \right) v = \frac{|v|^2}{r} = \frac{1}{r},
\]
for any unit vector \( v \), it follows that the principal curvatures are both \( r^{-1} \).

**Example.** Consider the 1-sheeted hyperboloid
\[
S = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = a^2 \}.
\]
This can be written in cylindrical coordinates as
\[
S = \{ (r \cos \theta, r \sin \theta, z) : r^2 - z^2 = a^2 \},
\]
where \( r^2 = x^2 + y^2 \). It is the level set \( f = a^2 \) for the function \( f(x, y, z) = x^2 + y^2 - z^2 \).

Therefore,
\[
H(e_1, e_1) = \frac{1}{\sqrt{a^2 + 2z^2}} \left[ \begin{array}{ccc} -\sin \theta & \cos \theta & 0 \\ 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{c} -\sin \theta \\ \cos \theta \\ 0 \end{array} \right] = \frac{1}{\sqrt{a^2 + 2z^2}} w
\]
\[
H(e_1, e_2) = H(e_2, e_1) = \frac{1}{a^2 + 2z^2} \left[ \begin{array}{ccc} z \cos \theta & z \sin \theta & r \\ 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{c} -\sin \theta \\ \cos \theta \\ 0 \end{array} \right] = \frac{1}{a^2 + 2z^2} \left[ \begin{array}{c} z \cos \theta \\ z \sin \theta \\ r \end{array} \right] = 0
\]
\[
H(e_2, e_2) = \frac{1}{(a^2 + z^2)^{3/2}} \left[ \begin{array}{ccc} z \cos \theta & z \sin \theta & r \\ 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{c} z \cos \theta \\ z \sin \theta \\ r \end{array} \right] = \frac{z^2 - r^2}{(a^2 + 2z^2)^{3/2}} = \frac{-a^2}{(a^2 + 2z^2)^{3/2}}.
\]
It follows that the principal curvatures are

\[ \frac{1}{\sqrt{a^2 + 2z^2}} \quad \text{and} \quad \frac{-a^2}{(a^2 + 2z^2)^{3/2}}. \]

**Example.** More generally, consider a surface of revolution

\[ S = \{(r(z) \cos \theta, r(z) \sin \theta, z)\}, \]

where \( r : \mathbb{R} \to (0, \infty) \) is the given radial profile function. This is the level set \( f = 0 \) of the function

\[ f(x, y, z) = \frac{1}{2}(x^2 + y^2 - (r(z))^2). \]

Therefore,

\[
\begin{align*}
\partial f &= \langle x, y, -rr_z \rangle \\
|\partial f| &= \sqrt{x^2 + y^2 + (rr_z)^2} \\
&= r\sqrt{1 + r_z^2} \\
\partial^2 f &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -rr_z - (r_z)^2 \end{bmatrix} \\
\frac{\partial^2 f}{|\partial f|} &= \frac{1}{r\sqrt{1 + r_z^2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -rr_z - (r_z)^2 \end{bmatrix} \\
&= \frac{1}{r\sqrt{1 + r_z^2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -rr_z - (r_z)^2 \end{bmatrix}.
\end{align*}
\]

An orthonormal basis of \( S \) is given by

\[
\begin{align*}
e_1 &= \frac{\langle -y, x, 0 \rangle}{\sqrt{x^2 + y^2}} \\
&= \frac{\langle -y, x, 0 \rangle}{r} \\
e_2 &= \frac{\langle r_zx, r_zy, r \rangle}{r\sqrt{1 + r_z^2}}.
\end{align*}
\]
Therefore, the second fundamental form of $S$ is given by

$$H(e_1, e_1) = \frac{1}{r^3\sqrt{1 + r_z^2}} \begin{bmatrix} -y & x & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -rr_{zz} - (r_z)^2 \end{bmatrix} \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

$$= \frac{1}{r\sqrt{1 + r_z^2}}$$

$$H(e_1, e_2) = H(e_2, e_1) = \frac{1}{r^3(1 + r_z^2)} \begin{bmatrix} r_zx & r_zy & r \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -rr_{zz} - (r_z)^2 \end{bmatrix} \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

$$= 0$$

$$H(e_2, e_2) = \frac{1}{r^3(1 + r_z^2)^{3/2}} \begin{bmatrix} r_zx & r_zy & r \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -rr_{zz} - (r_z)^2 \end{bmatrix} \begin{bmatrix} r_zx \\ r_zy \\ r \end{bmatrix}$$

$$= \frac{(rr_z)^2 - r^2(rr_{zz} + (r_z)^2)}{r^3(1 + r_z^2)^{3/2}}$$

$$= \frac{-r_{zz}}{(1 + r_z^2)^{3/2}}.$$
Appendix A

Notation

A.1 Cartesian spaces

The $m$-dimensional Cartesian affine space will be denoted $\mathbb{R}^m$, and a point in it by

$$x = (x^1, \ldots, x^m) = \begin{pmatrix} x^1 \\ \vdots \\ x^m \end{pmatrix} \in \mathbb{R}^m.$$ 

Its tangent space will be denoted $\mathbb{R}^m = T_0\mathbb{R}^m$, and a vector in it by

$$v = v^i \partial_i = \langle v^1, \ldots, v^m \rangle = [v^1, \ldots, v^m]^t = \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix} \in \mathbb{R}^m.$$ 

In particular, the standard basis of $\mathbb{R}^m$ is $[\partial_1, \ldots, \partial_m]$, where, for each $1 \leq i \leq m$,

$$\partial_i = [0, \ldots, 1, \ldots, 0]^t,$$ 

where the 1 is in the $i$-th slot.

The cotangent space (dual to $\mathbb{R}^m$) will be denoted $\mathbb{R}^m = T_0^*\mathbb{R}^m$, and a covector in it by

$$\theta = b_i dx^i = \langle b_1, \ldots, b_m \rangle^* = [b_1, \ldots, b_m] = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} \in \mathbb{R}^m.$$ 

In particular, the basis dual to $[\partial_1, \ldots, \partial_m]$ is denoted

$$[dx^1, \ldots, dx^m]^t = \begin{bmatrix} dx^1 \\ \vdots \\ dx^m \end{bmatrix}.$$ 

A.2 Abstract vector space

A basis of an $m$-dimensional abstract vector space will be denoted

$$E = [e_1, \ldots, e_m] = [e_1 \cdots e_m],$$
and a vector with respect to this basis as

\[ v = e_i a^i = [e_1 \cdots e_m] \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}. \]

The dual basis to \( E \) is denoted

\[ \Theta = [\theta^1, \ldots, \theta^m]^t = \begin{bmatrix} \theta^1 \\ \vdots \\ \theta^m \end{bmatrix}, \]

and a covector with respect to this basis by

\[ \omega = b_i \theta^i = [b_1 \cdots b_m] \begin{bmatrix} \theta^1 \\ \vdots \\ \theta^m \end{bmatrix}. \]

\section*{A.3 Manifold}

Coordinates on an \( m \)-dimensional manifold \( M \) is a \( C^1 \) map

\[ x = (x^1, \ldots, x^m) : M \to \mathbb{R}^m \]

that is injective and

\[ [dx^1, \ldots, dx^m]^t = \begin{bmatrix} dx^1 \\ \vdots \\ dx^m \end{bmatrix} \]

is a basis of \( T_p^* M \), for each \( p \in M \),

where, for each \( 1 \leq i \leq m \), \( dx^i \) is the differential of the \( C^1 \) function \( x^i : M \to \mathbb{R} \).

The dual basis at each \( p \in M \) is a basis of the tangent space \( T_p M \) and is denoted by

\[ [\partial_1, \ldots, \partial_m]. \]

This can also be defined equivalently as follows: The map \( x : M \to \mathbb{R}^m \) is a diffeomorphism onto its image, and therefore, the inverse map

\[ x^{-1} : x(M) \to M \]

is a \( C^1 \) diffeomorphism. For each \( p \in M \), there are \( m \) coordinate curves passing through it, where each coordinate curve is defined by holding all but one of the coordinates fixed and letting the remaining coordinate be the curve parameter. The velocity vectors of these curves at \( p \) are the tangent vectors \( \partial_1, \ldots, \partial_m \).

Note that not every manifold has coordinates. However, a manifold is always the union of manifolds that have coordinates.
A.4 Abstract affine space

Associated to each affine space $\mathbb{A}^m$ is its tangent space $\hat{\mathbb{A}}^m$. Recall that each affine basis $P = (p_0, \ldots, p_m)$ of $\hat{\mathbb{A}}^m$ defines a map

$$I_P : \mathbb{R}^m \to \mathbb{A}^m$$

$$[a^1, \ldots, a^m]^t \mapsto p_0 + a^iv_i,$$

where

$$v_i = p_i - p_0,$$

for each $1 \leq i \leq m$.

The map

$$x_P : \hat{\mathbb{A}}^m \to \mathbb{R}^m$$

$$p \mapsto 0 + I_{\hat{P}}^{-1}(p)$$

defines coordinates on $\mathbb{A}^m$, known as affine coordinates with respect to the affine basis $P$. 
Appendix B

Normal forms of maps

B.1 Normal form of a linear map

Lemma B.1. Let $L : \mathbb{V}^m \to \mathbb{W}^n$ be a linear map. There exist invertible linear maps $A : \mathbb{R}^m \to \mathbb{V}^m$ and $B : \mathbb{R}^n \to \mathbb{W}^n$ such that

$$B^{-1}LA = \begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix},$$

where $I_r$ is the $r$-by-$r$ identity matrix and $0_{k,l}$ is the $k$-by-$l$ zero matrix.

It follows that the maximum possible rank of a linear map $L : \mathbb{V}^m \to \mathbb{W}^n$ is $\min(m, n)$. If $m \geq n$ and $L : \mathbb{V}^m \to \mathbb{W}^n$ has maximum rank, then the rank is $n$,

$$L = B \begin{bmatrix} I_n & 0_{n,m-n} \end{bmatrix} A^{-1},$$

and therefore $L$ is surjective.

If $m \leq n$ and $L : \mathbb{V}^m \to \mathbb{W}^n$ has maximum rank, then the rank is $m$,

$$L = B \begin{bmatrix} I_m \\ 0_{n-m,m} \end{bmatrix} A^{-1},$$

and therefore $L$ is injective.

It follows that if $m = n$ and $L : \mathbb{V}^m \to \mathbb{W}^n$ has maximum rank, then

$$L = B \begin{bmatrix} I_m \end{bmatrix} A^{-1}$$

is bijective.

B.2 Characterization of a linear subspace

Lemma B.2. Given $0 \leq m \leq n$, an $n$-dimensional vector space $\mathbb{W}^m$, and $S \subset \mathbb{R}^n$, the following are equivalent:

1. $S$ is an $m$-dimensional linear subspace of $\mathbb{W}^m$.
APPENDIX B. NORMAL FORMS OF MAPS

2. There exists a linear map $L : \mathbb{R}^m \to \mathbb{W}^n$ with rank $m$ such that
   \[ \text{im } L = S. \]

3. There exists a linear map $M : \mathbb{W}^m \to \mathbb{R}^{n-m}$ with rank $n - m$ such that
   \[ \text{ker } M = S. \]

The most basic example is
\[ S = \{(x^1, \ldots, x^m, 0, \ldots, 0) : (x^1, \ldots, x^m) \in \mathbb{R}^m\} \subset W = \mathbb{R}^n, \]
with the linear maps
\[
L_{n,m} : \mathbb{R}^m \to \mathbb{R}^n \\
(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0)
\]
and
\[
M_{n,m} : \mathbb{R}^n \to \mathbb{R}^{n-m} \\
(x^1, \ldots, x^n) \mapsto (x^{n-m+1}, \ldots, x^n).
\]

**Lemma B.3.** If $S \subset \mathbb{W}^n$ is an $m$-dimensional subspace, then there exists a basis $E$ of $\mathbb{W}^n$
such that the linear maps $L = I_EL_{n,m}$ and $M = M_{n,m}I_E^{-1}$ satisfy the properties in Lemma B.2.

### B.3 Linear subspace as a graph

A 1-dimensional linear subspace of $\mathbb{R}^2$ is given by an equation of the form
\[ ax + by = 0. \]

When can this be written as a graph of a function, $y = f(x)$? The line cannot be vertical. Equivalently, $b$ must be nonzero, and therefore we can solve for $y$,
\[ y = -\frac{a}{b}x. \]

A 2-dimensional linear subspace of $\mathbb{R}^3$ is given by an equation of the form
\[ ax + by + cz = 0. \]

Similarly, this is the graph of a function, if $c \neq 0$ and therefore we can solve for $z$
\[ z = -\frac{a}{c}x - \frac{b}{c}y. \]

A 2-dimensional linear subspace of $\mathbb{R}^4$ is given by two linear equations of the form
\[
\begin{align*}
  a_1^1 x^1 + a_1^2 x^2 + a_1^3 x^3 + a_1^4 x^4 &= 0 \\
  a_2^1 x^1 + a_2^2 x^2 + a_2^3 x^3 + a_2^4 x^4 &= 0
\end{align*}
\]
When can we solve for \((x^3, x^4)\) in terms of \((x^1, x^2)\)? First, note that the system can be rewritten in the form
\[
\begin{bmatrix} A' & A'' \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = 0,
\]
(B.1)

where
\[
A' = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{bmatrix},
A'' = \begin{bmatrix} a_3^1 & a_3^2 \\ a_4^1 & a_4^2 \end{bmatrix},
\]
\[
x' = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix},
x'' = \begin{bmatrix} x^3 \\ x^4 \end{bmatrix}.
\]

We can therefore solve for \(x' = (x^3, x^4)\) if and only if the matrix \(A''\) is invertible. If so, we can multiply (B.1) on the left by \((A'')^{-1}\) to get an equivalent system of the form
\[
\begin{bmatrix} (A'')^{-1}A' & I \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = 0,
\]

which expands to
\[
\begin{align*}
b_1^1 x^1 + b_1^2 x^2 + x^3 &= 0 \\
b_2^1 x^1 + b_2^2 x^2 + x^4 &= 0,
\end{align*}
\]

where
\[
\begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} = (A'')^{-1}.
\]

Recall that \(\text{Hom}(\mathbb{R}^m, \mathbb{R}^n)\) is the space of \(n\)-by-\(m\) matrices. In general, an \(m\)-dimensional subspace of \(\mathbb{R}^n\), where \(m < n\), is given by a linear system
\[
Lx = \begin{bmatrix} L' & L'' \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix}, = 0,
\]
(B.2)

where \(x' \in \mathbb{R}^m, x'' \in \mathbb{R}^{n-m}, L' \in \text{Hom}(\mathbb{R}^m, \mathbb{R}^{n-m}),\) and \(L'' \in \text{Hom}(\mathbb{R}^{n-m}, \mathbb{R}^{n-m})\). This system can be rewritten equivalently as
\[
x'' = Ax',
\]
if and only if \(L''\) is invertible. Multiplying (B.2) on the left by \((L'')^{-1}\), we get an equivalent system
\[
\begin{bmatrix} (L'')^{-1}L' & I \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = 0,
\]

and therefore, we can solve for \(x''\) in terms of \(x'\):
\[
x'' = -(L'')^{-1}L'x'.
\]
APPENDIX B. NORMAL FORMS OF MAPS

B.4 Topology of the space of linear maps

Recall that, if we choose a basis for a vector space $V^m$ and one for another vector space $W^n$, then there is a linear isomorphism

$$\text{Hom}(V^m, W^n) \simeq \text{Hom}(\mathbb{R}^m, \mathbb{R}^n).$$

There is a natural inner product defined on $\text{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ given by

$$\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} A^i_j B^i_j,$$

which defines a norm, which in turn defines a distance function. We can therefore define open sets, convergence and limits of sequences, and continuity of functions and maps using the distance function. These definitions can then be transferred to the more abstract space $\text{Hom}(V^m, W^n)$.

Let $\text{GL}(V^m) \subset \text{Hom}(V^m, V^m)$ denote the space of invertible linear maps. For convenience we denote $\text{Hom}(\mathbb{R}^m, \mathbb{R}^m)$ by $\text{gl}(m)$ and $\text{GL}(\mathbb{R}^m)$ by $\text{GL}(m)$.

Recall that there is a norm defined on $\text{Hom}(\mathbb{R}^m, \mathbb{R}^m)$.

**Theorem B.4.** The following hold:

1. $\text{GL}(V^m)$ is an open subset of $\text{Hom}(V^m, V^m)$.

2. The map $F : \text{GL}(m) \rightarrow \text{GL}(m)$, where $F(L) \rightarrow L^{-1}$ is $C^\infty$.

3. The differential of $F$ is given by

$$dF(L)\dot{L} = -L^{-1}\dot{L}L^{-1}.$$

**Proof.** The determinant of a matrix is a continuous function of the matrix. It suffices to prove this for $V^m = \mathbb{R}^m$. First, we show that there exists $\delta > 0$ such that, for any $C \in \text{gl}(m)$,

$$|C| < \delta \implies I + C \in \text{GL}(m).$$

From this it follows that, given $A \in \text{GL}(m)$, there exists $\delta_A > 0$ such that for any $B \in \text{gl}(m)$,

$$|B| < \delta_A \implies A + b \in \text{GL}(m).$$

This shows that $\text{GL}(m)$ is an open subset of $\text{gl}(m)$.

Recall that the inverse of $A \in \text{GL}(m)$ is given by

$$A = \frac{1}{\det A} A^c,$$

where $A^c$ is the cofactor matrix of $A$. Since both $\det A$ and the components of $A^c$ are polynomial function of the components of $A$ and $\det A \neq 0$ for every $A \in \text{GL}(m)$, it follows that

$\square$
B.5 Local versus global behavior of a 1-dimensional nonlinear map

Compare the following maps $F : \mathbb{R}^1 \to \mathbb{R}^1$:

- $F_0(x) = x$
- $F_1(x) = x^3$
- $F_2(x) = x^2$
- $F_3(x) = \sin x$

The maps $F_0$ and $F_1$ have inverse maps

- $F_0^{-1}(y) = y$
- $F_1^{-1}(y) = y^{1/3}$,

but $F_1$ is not $C^1$ at $x = 0$. The map $F_2$ and $F_3$ have inverse maps, only if their domains are suitably restricted: If $\widehat{F}_2$ is $F_2$ restricted to $[0, \infty)$, then it has the inverse map $\widehat{F}_2^{-1} : [0, \infty) \to [0, \infty)$, given by

$\widehat{F}_2^{-1}(y) = \sqrt{y}$,

which is $C^1$ if restricted to $(0, \infty)$. If $\widehat{F}_3$ is $F_3$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then it has the inverse map $\widehat{F}_3 = \arcsin$, which is $C^1$ if restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

We shall focus on determining whether a map is locally invertible and the inverse is $C^1$. In other words, given $x_0$ in the domain of a map $F$, is there an open set $O$ containing $x_0$ such that $F$ restricted to $O$ has a $C^1$ inverse map.

B.6 Normal form of $C^1$ maps with maximal rank

Definition B.5. If $O \subset \mathbb{V}^m$ and $F : O \to \mathbb{W}^n$ is a $C^1$ map, then $F$ has maximal rank at $p \in O$, if its differential at $p \in O$, $dF(p) : \mathbb{V}^m \to \mathbb{W}^n$ has maximal rank.

A $C^1$ map $F : O \subset \mathbb{V}^m$ has maximal rank on $O$, if it has maximal rank at every $p \in O$.

Definition B.6. If $m \geq n$ and $F$ has maximal rank, then $F$ is a submersion of $O$.

If $m \leq n$ and $F$ has maximal rank, then $F$ is an immersion of $O$.

If $m = n$ and $F$ has maximal rank, then $F$ is a local diffeomorphism of $O$.

Definition B.7. Let $O \subset \mathbb{R}^m$ be open. A map $F : O \to \mathbb{R}^m$ is a $C^1$ diffeomorphism, if $F(O)$ is open, $F$ is $C^1$, and there exists a $C^1$ map $G : F(O) \to O$ such that

- $G(F(x)) = x$, $\forall \ x \in O$
- $F(G(y)) = y$, $\forall \ y \in F(O)$.

In particular, if the map $G$ exists, it is unique and the maps $F$ and $G$ are bijective.

If $F : O \to \mathbb{R}^n$ is a $C^1$ map, it is a local diffeomorphism at $x_0 \in O$, if there exists an open neighborhood $O' \subset O$ of $x_0$ such that $F$ restricted to $O'$ is a $C^1$ diffeomorphism.

By the chain rule,

Lemma B.8. If $F : O \to \mathbb{R}^n$ is a $C^1$ diffeomorphism at $x_0 \in O$, then the differential $dF(x) : \mathbb{R}^m \to \mathbb{R}^n$ is invertible for any $x \in O$. 
APPENDIX B. NORMAL FORMS OF MAPS

B.7 Contraction map lemma

We state this for an arbitrary vector space with a norm with respect to which the space is a complete topological space. In other words, any Cauchy sequence has a limit.

Lemma B.9. Let $\mathbb{V}$ be a complete normed vector space and $O \subset \mathbb{V}$ an open subset. If $F : O \rightarrow O$ is a continuous map such that there exists $0 \leq c < 1$ such that for any $x_0, x_1 \in O$,

$$|F(x_1) - F(x_0)| \leq c|x_1 - x_0|,$$

then there exists a unique $x \in O$ such that $F(x) = x$.

Proof. Given $x_0 \in O$, define a sequence as follows: For each $k \geq 0$, let

$$x_{k+1} = F(x_k).$$

It follows that, for each $k \geq 1$,

$$|x_{k+1} - x_k| = |F(x_k) - F(x_{k-1})| \leq c|x_k - x_{k-1}|.$$

From this, it follows by induction that, for any $0 \leq k \leq l$,

$$|x_{l+1} - x_l| \leq c^{l-k}|x_{k+1} - x_k|.$$

Therefore,

$$|x_l - x_k| \leq |x_l - x_{l-1}| + \cdots + |x_{k+1} - x_k|$$

$$(c^{l-k} + \cdots + 1)|x_{k+1} - x_k| \leq \frac{1 - c^{l-k+1}}{1 - c}|x_{k+1} - x_k|$$

$$\leq c^k|x_1 - x_0|.$$

For $\epsilon < 0$, let $N > 0$ satisfy

$$c^N|x_1 - x_0| = \epsilon.$$

Then for any $N \leq k \leq l$,

$$|x_l - x_k| \leq c^k|x_1 - x_0| \leq c^N|x_1 - x_0| \leq \epsilon.$$

It follows that $(x_1, \cdots)$ is a Cauchy sequence and has a limit $x \in O$. By the continuity of $F$

$$x = \lim_{k \rightarrow \infty} x_{k+1} = \lim_{k \rightarrow \infty} F(x_k) = F(x).$$

Finally, if $F(x_1) = x_1$ and $F(x_2) = x_2$, then

$$|x_2 - x_1| = |F(x_2) - F(x_1)| \leq c|x_2 - x_1|$$

which implies that

$$(1 - c)|x_2 - x_1| \leq 0,$$

and therefore $x_2 = x_1$. \qed
### B.8 Inverse function theorem

The following is a version of the mean value theorem.

**Lemma B.10.** If $F : O \to \mathbb{R}^n$ is a $C^1$ map and $x_0, x_1 \in O$ are such that the line segment joining them also lies in $O$, then

$$F(x_1) - F(x_0) = \int_0^1 dF'((1 - t)x_0 + tx_1)(x_1 - x_0) \, dt.$$

**Proof.**

$$F(x_1) - F(x_0) = \int_0^1 \frac{d}{dt} F((1 - t)x_0 + t x_1) \, dt$$

$$= \int_0^1 dF'((1 - t)x_0 + tx_1)(x_1 - x_0) \, dt.$$

The following is a converse to Lemma B.8:

**Lemma B.11.** Given $A \in \text{GL}(m)$ and $v \in \mathbb{R}^m$, $|Av| \leq |A||v|$.

**Theorem B.12.** If $F : O \to \mathbb{R}^m$ is a $C^1$ map and $x_0 \in O$, then if $dF(x_0) : \mathbb{R}^m \to \mathbb{R}^m$ is an invertible linear map, then there exists an open neighborhood $O' \subset O$ of $x_0$ such that $F$ restricted to $O'$ is a $C^1$ diffeomorphism. In particular, there $F(O')$ is open and there exists a $C^1$ map $G : F(O') \to O$ such that

$$F(G(y)) = y, \quad \forall \ y \in F(O') \text{ and } G(F(x)) = x, \quad \forall \ x \in O'.$$

**Proof.** Let $L : \mathbb{R}^m \to \mathbb{R}^m$ be the inverse linear map to $dF(x_0)$ and

$$O' = B(x_0, \delta),$$

where

$$\delta = \frac{1}{2} |L|^{-1}.$$

Given $y \in F(O')$, if

$$\Phi(x) = x - L(y - F(x)),$$

then $y = F(x)$ if and only if $\Phi(x) = x$, so it suffices to show that $\Phi : O' \to O'$ is a contraction mapping. First, note that, for any $x_1, x_2 \in O'$,

$$\Phi(x_2) - \Phi(x_1) = \int_0^1 dF((1 - t)x_2 + tx_1)(x_2 - x_1) \, dt$$

$$x_2 - x_1 = LdF(x_0)(x_2 - x_1)$$
and therefore
\[
\Phi(x_2) - \Phi(x_1) = LdF(x_0)(x_2 - x_1) + L^{-1}(F(x_2) - F(x_1))
\]
\[
= L \int_0^1 (dF(x_0) - dF((1-t)x_2 + tx_1))(x_2 - x_1) \, dt
\]
It follows that
\[
|\Phi(x_2) - \Phi(x_1)| \leq |L| \frac{1}{2} |L|^{-1} |x_2 - x_1|
\]
\[
\leq \frac{1}{2} |x_2 - x_1|.
\]

\[\square\]

B.9 Normal forms of nonlinear maps

Definition B.13. Given an open set \( O \subset \mathbb{R}^m \) and \( m \leq n \), a \( C^1 \) map \( F : O \to \mathbb{R}^n \) is an immersion, if for any \( x \in O \), the differential \( dF(x) : \mathbb{R}^m \to \mathbb{R}^n \) is injective. This is equivalent to saying that \( m \leq n \) and the rank of \( dF(x) \) is \( m \).

If \( F \) is also injective, then it is an embedding.

Theorem B.14. If \( F : O \to \mathbb{R}^n \) is a \( C^1 \) immersion, then for each \( x \in O \), there exists an open neighborhood \( O' \subset O \) of \( x \) and diffeomorphisms \( A : A^{-1}(O') \to O' \) and \( B : B^{-1}(F(O')) \to F(O') \) such that
\[
F = B \circ L_{n,m} \circ A^{-1}
\]

Definition B.15. Given an open set \( O \subset \mathbb{R}^n \), a \( C^1 \) map \( F : O \to \mathbb{R}^{n-m} \) is a submersion, if for any \( x \in O \), the differential \( dF(x) : \mathbb{R}^n \to \mathbb{R}^{n-m} \) has rank \( n - m \). This is equivalent to \( dF(x) \) being surjective.

Theorem B.16. If \( F : O \to \mathbb{R}^{n-m} \) is a \( C^1 \) submersion, then for each \( x \in O \), there exists an open neighborhood \( O' \subset O \) of \( x \) and diffeomorphisms \( A : A^{-1}(O') \to O' \) and \( B : B^{-1}(F(O')) \to F(O') \) such that
\[
F = B \circ M_{n,m} \circ A^{-1}
\]
Appendix C

The differential of a function

C.1 The dual vector space

**Definition C.1.** The dual vector space of a vector space $V^m$ is defined to be the space of all linear functions of $V^m$,

$$(V^m)^* = \{ \ell : V^m \to \mathbb{R} : \ell \text{ is linear} \}.$$  

Observe that $(V^m)^*$ is indeed a vector space. If $v_1, \ldots, v_m$ is a basis of $V^m$, define $\ell_1, \ldots, \ell_m \in V^*$ as follows:

$$\ell_i(a_1v^1 + \cdots + a_mv^m) = a_i.$$  

Equivalently,

$$\ell_i(v^j) = \delta^j_i, \ \forall 1 \leq i, j \leq m.$$  

**Lemma C.2.** $\ell_1, \ldots, \ell_m$ is a basis of $(V^m)^*$ and therefore $\dim(V^m)^* = \dim V^m$.

**Lemma C.3.** There is a natural isomorphism

$$V^m \to (V^m)^*$$

that maps $v \in V^m$ to the linear function on $(V^m)^*$ given by

$$\ell \mapsto \ell(v).$$  

Therefore, for any $v \in V^m$ and $\ell \in (V^m)^*$,

$$v(\ell) = \ell(v).$$  

where on the left $v$ is treated as an element of $(V^m)^*$ and on the right as an element of $V$. Because of this symmetry, we will denote this by

$$\langle \ell, v \rangle = \langle v, \ell \rangle = \ell(v).$$
C.2 Differentiability of a function

Definition C.4. Let $O \subseteq \mathbb{A}^m$ be open. A function $f : O \to \mathbb{R}$ is differentiable at $p \in O$, if there exists a linear function $\ell : \mathbb{V}^m \to \mathbb{R}$ (which depends on $f$ and $p$) such that
\[
\lim_{t \to 0} \frac{f(p + tv) - f(p) - t\langle \ell, v \rangle}{t} = 0.
\]
Equivalently, there exists a real-valued function $\epsilon_f(p, v, t)$ such that
\[
f(p + tv) = f(p) + t\langle \ell, v \rangle + t\epsilon(p, v, t)
\]
\[
\lim_{t \to 0} \epsilon(p, tv) = 0.
\]

C.3 Directional derivative of a function

Definition C.5. Given $f : O \to \mathbb{R}, p \in O$, and $v \in \mathbb{V}^m$, the directional derivative of $f$ at $p$ in the direction $v$ is, if it exists, defined to be
\[
d_v f(p) = \left. \frac{d}{dt} \right|_{t=0} f(p + tv) = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t}.
\]
In this definition the function $v \mapsto d_v f(p)$ is not assumed to be linear. In fact,

Lemma C.6. The function $f : O \to \mathbb{R}$ is differentiable at $p \in O$ if and only if the function $v \mapsto d_v f(p)$ exists for all $v \in \mathbb{V}^m$ and is linear.

C.4 $C^1$ functions

Definition C.7. A function $f : O \to \mathbb{R}$ is $C^1$ on $O$, if, for each $v \in \mathbb{V}^m$, $d_v f : O \to \mathbb{R}$
\[
p \mapsto d_v f(p)
\]
is a continuous function on $O$.

Lemma C.8. A $C^1$ function on $O \subseteq \mathbb{A}^m$ is differentiable on $O$.

Since, if $f$ is $C^1$, $v \mapsto d_v f(p) \in \mathbb{R}$ is linear, we can define $df(p) \in (\mathbb{V}^m)^*$ to be the linear function, where
\[
\langle df(p), v \rangle = d_v f(p).
\]
This in turn defines a map $df : O \to (\mathbb{V}^m)^*$, which is called the differential of $f$.

C.5 Differential 1-form on an open subset of $\mathbb{A}^m$

Definition C.9. Given an open set $O \subseteq \mathbb{A}^m$, a differential 1-form is a map $\omega : O \to \mathbb{V}^*$.

Example. The differential of a function is a 1-form.
C.6  Line integral of a 1-form along a curve in $\mathbb{A}^m$

**Definition C.10.** If $O \subset \mathbb{A}^m$ is open, $c : [0, T] \to O$ is a $C^1$ curve, and $\omega : O \to (\mathbb{V}^m)^*$ a 1-form, then the line integral of $\omega$ along $c$ is defined to be

$$\int_c \omega = \int_0^T \langle \omega(c(t)), c'(t) \rangle \, dt.$$  

Observe that the integrand on the right is a real-valued function of $t \in [0, T]$ and therefore the integral is standard single variable integral.

By using the same proof of the chain rule for functions of a single variable, the following holds:

**Lemma C.11.** If $c : I \to \mathbb{A}^m$ is a $C^1$ curve and $\phi : I' \to I$ is a $C^1$ function on the interval $I'$, then

$$(c \circ \phi)'(t) = c'(\phi(t))\phi'(t).$$

The same proof for the change of variables for a single variable integral now implies:

**Theorem C.12.** If $O \subset \mathbb{A}^m$ is open, $c : [0, T] \to O$ is a $C^1$ curve, and $\omega : O \to (\mathbb{V}^m)^*$ a 1-form, then for any $C^1$ function $\phi : [0, \hat{T}] \to [0, T]$ such that $\phi(0) = 0$ and $\phi(\hat{T}) = T$,

$$\int_{\hat{c}} \omega = \int_c \omega,$$

where $\hat{c} = c \circ \phi$.

The fundamental theorem of calculus implies the following:

**Theorem C.13.** If $c : [0, T] \to \mathbb{A}^m$ is a $C^1$ curve and $f : O \to \mathbb{R}$ a $C^1$ function on an open set $O \subset \mathbb{A}^m$, then

$$\int_c df = f(c(T)) - f(c(0)).$$

In particular, the line integral of $df$ from $p$ to $q$ does not depend on which $C^1$ curve is used to connect $p$ to $q$.

C.7  Fundamental example

Recall that, associated with an affine basis $(p_0, \ldots, p_m)$ of $\mathbb{A}^m$ is a basis $E = (e - 1, \ldots, e_m)$ of $\mathbb{V}^m$, where

$$e_1 = p_1 - p_0, \ldots, e_m = p_m - p_0.$$  

On the other hand, recall that $I_{E^{-1}}(p) = (x^1, \ldots, x^m)$, where for each $1 \leq i \leq m$, the

$$x^i : \mathbb{A}^m \to \mathbb{R}$$

$$p_0 + v \mapsto v^i.$$
The differential of $x^i$ is defined to be $dx^i \in (\mathbb{V}^m)^*$, where

$$
\langle dx^i(p), v \rangle = \frac{d}{dt} \bigg|_{t=0} x^i(p + tv) = v^i.
$$

Equivalently,

$$
\langle dx^i, e_j \rangle = \delta_j^i.
$$

and, therefore, $(dx^1, \ldots, dx^m)$ is the basis of $(\mathbb{V}^m)^*$ dual to the basis $E = (e_1, \ldots, e_m)$ of $\mathbb{V}^m$.

Given a $C^1$ function $f : O \to \mathbb{R}$, note that, if $p_0 \in O$ and $I_E(x^1, \ldots, x^n) = p$, then

$$
\partial_i((f \circ I_E)(x^1, \ldots, x^n)) = \frac{d}{dt} \bigg|_{t=0} f(p + te_i) = \langle df(p), e_i \rangle.
$$

Therefore,

$$
df(p) = \langle df(p), e_i \rangle dx^i.
$$

This leads to the following notation: Given an affine basis $P = (p_0, \ldots, p_m)$ of $\mathbb{A}^m$, instead of denoting the corresponding basis of $\mathbb{V}^m$ by $E = (e_1, \ldots, e_m)$, we denote it by $\partial = (\partial_1, \ldots, \partial_m)$, where, for each $i = 1, \ldots, m$,

$$
\partial_i = p_i - p_0.
$$

As noted above, its dual basis is $dx = (dx^1, \ldots, dx^m)$. Given a $C^1$ function $f : O \to \mathbb{R}$, we denote

$$
\partial_i f(p) = \langle df(p), \partial_i \rangle.
$$

Therefore, omitting the $p$ for brevity,

$$
df = \partial_i f \, dx^i.
$$
Appendix D

Differential of a map

D.1 Characterization of a linear map

Lemma D.1. Given a map $F : \mathbb{V}^m \to \mathbb{W}^n$, the following are equivalent:

- $F$ is linear.
- $\ell \circ F : \mathbb{V}^m \to \mathbb{R}$ is linear for any $\ell \in (\mathbb{W}^n)^*$.
- Given a basis $\ell^1, \ldots, \ell^n$ of $\mathbb{W}^n$, $\ell_i \circ F$ is linear for each $i = 1, \ldots, n$.

D.2 Norm of a linear map

Recall that given bases $E$ of $\mathbb{V}^m$ and $F$ of $\mathbb{W}^n$, then there is a linear isomorphism $I_{E,F} : M(n,m) \to \text{Hom}(\mathbb{V}^m, \mathbb{W}^n)$. Moreover, we can define a dot product on $M(n,m)$ given by

$$A \cdot B = \sum_{i=1}^{m} \sum_{j=1}^{n} A^i_j B^i_j$$

with the corresponding norm given by $|A|^2 = A \cdot A$. This is the same norm as if we simply flattened each matrix into an $mn$-dimensional vector.

D.3 Differential of a map

If $\mathbb{A}^m$ and $\mathbb{B}^n$ are affine spaces with tangent spaces $\mathbb{V}^m$ and $\mathbb{W}^n$ respectively, $O$ is an open subset of $\mathbb{A}^m$, then a map $F : O \to \mathbb{B}^n$ is differentiable at $p$, if there exists a linear map $L : \mathbb{V}^m \to \mathbb{W}^n$ such that, for any $v \in \mathbb{V}^m$,

$$\lim_{t \to 0} \frac{F(p + tv) - F(p) - tL(v)}{t} = 0.$$ 

If $F$ is differentiable at $p$, then the map $L : \mathbb{V}^m \to \mathbb{W}^n$ is called the differential of $F$ at $p$ and denoted $dF_p$. If $F$ is differentiable for every $p \in O$ and the map

$$dF : O \to \text{Hom}(\mathbb{V}^m, \mathbb{W}^n)$$

is continuous, then the map $F$ is $C^1$. 

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D.4 Chain Rule

Lemma D.2. If $F : \mathbb{A}^m \to \mathbb{B}^n$ and $G : \mathbb{B}^n \to \mathbb{C}$ are $C^1$ maps, then so is $G \circ F$, and

$$d(G \circ F)(p) = dG(F(p)) \circ dF(p).$$
Appendix E

Topology

E.1 Topology of \( \mathbb{R}^n \)

The topology of \( \mathbb{R}^n \) is described using the distance function and the concepts of open and closed sets. This is used to define the convergence of sequences and functions, which in turn are used to define differentiation and integration.

E.1.1 Dot product and norm of a vector

Recall that the dot product of \( \mathbf{v} = (v^1, \ldots, v^n), \mathbf{w} = (w^1, \ldots, w^n) \in \mathbb{R}^n \) is defined to be

\[
\mathbf{v} \cdot \mathbf{w} = v^1 w^1 + \cdots + v^n w^n
\]

and the norm of \( v \in \mathbb{R}^n \) is \( |v| \geq 0 \), where

\[
|v|^2 = \mathbf{v} \cdot \mathbf{v}.
\]

**Lemma E.1.** For any vectors \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \),

\[
|\mathbf{v} - \mathbf{w}|^2 \leq |\mathbf{v}|^2 + |\mathbf{w}|^2,
\]

with equality holding if and only if \( \mathbf{w} = t \mathbf{v} \), for some \( t \geq 0 \).

E.1.2 Distance between two points

Recall that the distance between two points \( p, q \in \mathbb{R}^n \) is defined to be \( d(p, q) \geq 0 \), where

\[
d(p, q)^2 = |p - q|^2 = (p - q) \cdot (p - q).
\]

The fundamental properties of the (or any) distance function are the following: For any points \( p, q, r \in \mathbb{R}^n \),

\[
\begin{align*}
d(p, q) &= d(q, p) \\
d(p, q) &\geq 0 \\
d(p, q) = 0 &\iff p = q \\
d(p, r) &\leq d(p, q) + d(q, r),
\end{align*}
\]
Denote the ball of radius \( r > 0 \) centered at \( p \in \mathbb{R}^n \) by
\[
B(p, r) = \{ x \in \mathbb{R}^n : d(p, x) < r \}
\]
and the closed ball by
\[
\overline{B}(p, r) = \{ x \in \mathbb{R}^n : d(p, x) \leq r \}
\]

E.1.3 Open and closed sets

Definition E.2. A set \( O \subset \mathbb{R}^n \) is open, if for any \( p \in O \), there exists \( r > 0 \) such that \( B(p, r) \subset O \). A set \( C \subset \mathbb{R}^n \) is closed, if \( \mathbb{R}^n \setminus C \) is open.

E.1.4 Convergent and Cauchy sequences

Definition E.3. A sequence is a map \( s : \mathbb{Z}_+ \rightarrow \mathbb{R}^n \).

Definition E.4. A sequence \( s_1, \ldots \) converges to the limit \( s_\infty \in \mathbb{R}^n \), if for any \( \epsilon > 0 \), there exists \( N(\epsilon) \) such that
\[
d(s_i, s_\infty) < \epsilon, \ \forall \ i \geq N(\epsilon).
\]

Definition E.5. A sequence is a Cauchy sequence, if for any \( \epsilon > 0 \), there exists \( N(\epsilon) \) such that
\[
d(s_i, s_j) < \epsilon, \ \forall \ i, j \geq N(\epsilon).
\]

Lemma E.6. A sequence converges to a limit if and only if it is Cauchy.

E.1.5 Compact sets

Definition E.7. Let \( S \subset \mathbb{R}^n \).

- \( S \) is bounded, if there exists \( R > 0 \) such that \( S \subset B(0, R) \).
- \( S \) is sequentially compact, if any sequence has a convergent subsequence.
- \( S \) is compact, if, given any infinite collection of open sets \( O_1, \ldots \) such that
\[
S \subset \bigcup_i O_i,
\]
there exists a finite subcollection \( O_1, \ldots, O_N \) such that
\[
S \subset O_1 \cup \cdots O_N.
\]

Lemma E.8. The following are equivalent for a set \( S \subset \mathbb{R}^n \):
1. \( S \) is compact.
2. \( S \) is sequentially compact.
3. \( S \) is closed and bounded.
E.2 Topology of affine space

It is less convenient to define directly the topology of affine space, because there is no natural
definition of distance.

Instead, we do it indirectly by the topology of $\mathbb{R}^m$. If $E = (e_0, \ldots, e_m)$ is an affine basis
of $\mathbb{A}^m$, then we can define a distance function $d_E$ on $\mathbb{A}^m$ using the distance function on $\mathbb{R}^m$
as follows: For each $p, q \in \mathbb{A}^m$,

$$d_E(p, q) = d(I_E(p), I_E(q)).$$

Although each affine basis defines a distance function, they are all equivalent in the following
sense:

**Lemma E.9.** If $E$ and $F$ are both affine bases of $\mathbb{A}^m$, then there exists $a, b > 0$ such that
for any $p, q \in \mathbb{A}^m$,

$$ad_E(p, q) \leq d_F(p, q) \leq bd_E(p, q).$$

This implies that a set in $\mathbb{A}^m$ is open with respect to the distance function $d_E$ if and only
if it is open with respect to $d_F$. It also implies the following:

**Lemma E.10.** A sequence $p_1, p_2, \ldots \in \mathbb{A}^m$ converges with respect to the distance function
$d_E$ if and only if it converges with respect to the distance function $d_F$, and the limit is the
same in both cases.

We will follow the following principle in what follows: When working on affine space,
do as much as possible abstractly without relying an affine basis or even an origin (which
defines an isomorphism of the affine space with its vector space). Even analysis arguments
that use the topology of affine space but do not require any explicit use of the distance
function can be done abstractly. However, when an analysis argument requires the use of a
distance function, use the distance function associated with an affine basis as defined above.
Appendix F

Maps

F.1 Maps on $\mathbb{R}^m$

F.1.1 Limits

Definition F.1. Given an open set $O \subset \mathbb{R}^m$, a map $f : O \rightarrow \mathbb{R}^n$, $x_0 \in O$, and $L \in \mathbb{R}^n$,

$$\lim_{x \to x_0} f(x) = L,$$

if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x_0, x) < \delta \implies d(f(x_0), f(x)).$$

If $n = 1$, we call $f$ a map.

F.1.2 Boundedness

A function $f : D \rightarrow \mathbb{R}$ is bounded from above, if there exists $U \in \mathbb{R}$ such that

$$f(x) \leq U, \; \forall x \in D.$$

Any $U$ satisfying this is an upper bound. A least upper bound is an upper bound $U_0 \in \mathbb{R}$ such that $U_0 \leq U$ for any upper bound $U$.

Lemma F.2. If a function $f : D \rightarrow \mathbb{R}$ is bounded from above, then there exists a least upper bound of $f$ on $D$.

A function $f : D \rightarrow \mathbb{R}$ is bounded from below, if there exists $L \in \mathbb{R}$ such that

$$L \leq f(x), \; \forall x \in D.$$

Any $U$ satisfying this is a lower bound. A greatest lower bound is an upper bound $L_0 \in \mathbb{R}$ such that $L_0 \geq L$ for any lower bound $L$.

A function is bounded, if it is bounded from both above and below.

The completeness of the real line implies the following:
Lemma F.3. Any function bounded from above has a unique least upper bound. Any function bounded from below has a unique greatest lower bound.

We can therefore denote the least upper bound of $f$ by
$$\sup_{x \in D} f(x),$$
and its greatest lower bound by
$$\inf_{x \in D} f(x).$$

F.1.3 Continuity

Given an open set $O \subset \mathbb{R}^m$, a map $f : O \to \mathbb{R}^n$ is continuous at $x_0 \in O$, if
$$\lim_{x \to x_0} f(x) = f(x_0).$$

A map $f : O \to \mathbb{R}^n$ is continuous at $x_0 \in O$, if
$$\lim_{x \to x_0} f(x) = f(x_0).$$

A map $f : O \to \mathbb{R}^n$ is continuous, if it is continuous at every $x_0 \in O$.

Lemma F.4. A map $f : O \to \mathbb{R}^n$ is continuous if and only if, for any open set $S \subset \mathbb{R}^n$, $f^{-1}(S)$ is an open subset of $O$.

Lemma F.5. If $D \subset \mathbb{R}^m$ is compact and $f : D \to \mathbb{R}^n$ is continuous, then it is bounded. Moreover, there exist $p, q \in D$ such that
$$f(p) = \inf_{x \in D} f(x) \text{ and } f(q) = \sup_{x \in D} f(x).$$

F.2 Maps on affine space

Let $P$ be an affine basis of the affine space $\mathbb{A}^m$, $I_P : \mathbb{R}^m \to \mathbb{A}^m$ be the affine isomorphism defined by $P$. Let $Q$ be an affine basis of the affine space $\mathbb{B}^n$ and $I_Q : \mathbb{R}^n \to \mathbb{B}^n$ be the affine isomorphism defined by $Q$. Let $O$ an open subset of $\mathbb{A}^m$.

Definition F.6. Given a map $f : O \to \mathbb{B}^n$, $p_0 \in O$, and $b \in \mathbb{B}^n$, $\lim_{p \to p_0} f(p) = b$

if and only if
$$\lim_{x \to x_0} f_{P,Q}(x) = b_Q,$$

where $x_0 = I_P^{-1}(p_0)$ and $f_{P,Q} = I_Q^{-1} \circ f \circ I_P(x)$, and $b_Q = I_Q^{-1}(b)$.

Definition F.7. A map $f : O \to \mathbb{B}^n$ is continuous if $I_Q^{-1} \circ f \circ I_P : I_E^{-1}(O) \to \mathbb{R}^n$ is continuous.
F.3 Lemmas

**Lemma F.8.** If \( O \subset \mathbb{A}^m \) is open, \( f, g : O \to \mathbb{R} \) are \( C^1 \) functions such that, for each \( p \in O \),

\[
    f(p) = 0 \implies \partial f(p) \neq 0 \quad \text{and} \quad g(p) = 0,
\]

then there exists a \( C^1 \) function \( \phi : O \to \mathbb{R} \) such that \( g = \phi f \). If \( f \) and \( g \) are \( C^2 \), then so is \( \phi \).

F.4 Partition of unity

Let \( M \) be an \( m \)-dimensional \( C^2 \) manifold. Therefore, there exists a countable locally finite covering of \( M \) by coordinate charts \( x_i : O_i \to B(0, 1) \subset \mathbb{R}^m \). In particular, each \( x_i \) is bijective and, for any \( i \) and \( j \), \( x_j \circ x_i^{-1} : x_i(O_i \cap O_j) \to x_j(O_i \cap O_j) \) is a \( C^2 \) diffeomorphism.

Let \( \phi : B(0, 1) \to [0, \infty) \) be given by

\[
    \phi(x) = \frac{1 + \cos(\pi |x|)}{2}.
\]

Let \( \phi_i : M \to [0, \infty) \) be the \( C^2 \) function given by

\[
    \psi_i(p) = \begin{cases} 
        \phi(x_i(p)) & \text{if } p \in O_i \\
        0 & \text{otherwise}
    \end{cases}
\]

Observe that

\[
    \psi = \sum_i \psi_i
\]

is strictly positive and therefore, the functions

\[
    \chi_i = \frac{\psi_i}{\psi}
\]

are well-defined \( C^2 \) functions on \( M \) such that

\[
    \sum_i \chi_i = 1.
\]
Bibliography
