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Optimal Sobolev Norms and the L^p Minkowski Problem

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Dedicated to Professor Rolf Schneider on the occasion of his sixty-fifth birthday

1 Introduction

Throughout this paper, $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$ ($n > 1$ is always assumed) that is normalized so that its unit ball has the same volume as the Euclidean unit ball.

A norm $\|\cdot\|$ on $\mathbb{R}^n$ induces a Sobolev norm for compactly supported functions $f : \mathbb{R}^n \to \mathbb{R}$ with L^1 weak derivative, given by

$$f \longmapsto \|\nabla f\|_1 = \int_{\mathbb{R}^n} \|\nabla f(x)\|_* dx, \tag{1.1}$$

where $\|\cdot\|_*$ denotes the norm dual to $\|\cdot\|$ (see Section 2 for precise definitions).

Cordero, Nazaret, and Villani [7] have recently used a beautiful mass transportation argument to establish the following family of sharp Gagliardo-Nirenberg inequalities for this Sobolev norm. If $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is compactly supported and smooth, then

$$\|\nabla f\|_1 \geq c_{1,r,n} |f|_1^{1-\alpha} |f|_r^{\alpha}, \tag{1.2}$$

where $0 < r \leq n/(n-1)$, $\alpha \in \mathbb{R}$ is determined by scale invariance, and $|f|_r$ denotes the

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standard L^r-norm of f. Their work extends earlier results of Maz′ja [25], Gromov [27], Alvino, Ferone, Trombetti, and Lions [1], and Del Pino and Dolbeault [8].

The CNV inequality immediately raises the obvious question.

The optimal L^1 *Sobolev norm.* Given a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^1 weak derivative, what is the unique norm $\|\cdot\|$ on $\mathbb{R}^n$ that minimizes $\|\nabla f\|_1$?

An apparently unrelated question is the following.

The even Minkowski problem. Given a positive even function g on the unit sphere S^{n-1}, what is the unique convex body K such that for each unit vector u, $g(u)$ is the Gauss curvature at the point on the boundary ∂K that has outer unit normal u?

One aim of this note is to show that the two questions stated above are essentially equivalent. We will in fact consider L^p-generalizations of these questions. This can be stated as follows (see Section 2 for precise definitions).

If $1 \le p < n$, a norm $\|\cdot\|$ on $\mathbb{R}^n$ induces a Sobolev norm for compactly supported functions $f : \mathbb{R}^n \to \mathbb{R}$ with L^p weak derivative, given by

$$f \longmapsto \|\nabla f\|_p = \left(\int_{\mathbb{R}^n} \|\nabla f(x)\|_*^p \, dx \right)^{1/p}, \tag{1.3}$$

where $\|\cdot\|_*$ denotes the dual norm.

Cordero, Nazaret, and Villani [7] extended earlier results of Aubin [2], Talenti [31], Gromov [27], Alvino, Ferone, Trombetti, and Lions [1], and Del Pino and Dolbeault [8] and established the following family of sharp L^p Gagliardo-Nirenberg inequalities (throughout this paper, they will be called the CNV *inequalities*): If $1 \le p < n$, $\|\cdot\|$ is a norm on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ is compactly supported and smooth, then

$$\|\nabla f\|_p \ge c_{p,r,n} |f|_q^{1-\alpha} |f|_r^{\alpha}, \tag{1.4}$$

where $0 < r \le np/(n-p)$, $q = 1 + r(p-1)/p$, $|f|_p$ denotes the standard L^p-norm of f, and $\alpha \in \mathbb{R}$ is determined by scale invariance.

This leads us to ask the following for every $p \ge 1$ (and not just $1 \le p < n$).

The optimal L^p *Sobolev norm.* Given a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^p weak derivative, what is the unique norm $\|\cdot\|$ on $\mathbb{R}^n$ that minimizes $\|\nabla f\|_p$?

In this paper, we show that this problem is essentially the same as the apparently unrelated even L^p Minkowski problem. The L^p Minkowski problem, which can be written as a Monge-Ampère equation

$$h^{1-p}\det\left(h_{ij}+h\delta_{ij}\right)=g \tag{1.5}$$

on the unit sphere, is a central question in the L^p Brunn-Minkowski theory of convex bodies (see Section 3 for more details).

A consequence of Theorem 5.1 in this paper is that all possible L^p Sobolev norms of a function $f:\mathbb{R}^n\to\mathbb{R}$ can be encoded naturally within a single origin-symmetric convex body K. In particular, for each norm on $\mathbb{R}^n$, the corresponding L^p Sobolev norm of f is given by the (normalized) L^p mixed volume of K and the unit ball of the norm (see Theorem 5.1 and the remark immediately following it for details).

Moreover, the (suitably normalized) volume of this convex body is precisely equal to the optimal L^p Sobolev norm of f. We show in Section 6 that minimizing the left-hand side of the CNV inequality (1.4) over all norms on $\mathbb{R}^n$ establishes an affine version of the Cordero-Nazaret-Villani inequalities.

Zhang [34] and the authors [22] have recently established a sharp L^p affine Sobolev inequality (a version of the L^1 affine Sobolev inequality involving capacity has recently been established by Xiao [33]). The proof in [22] is rather involved, using the L^p Petty projection inequality established by the authors [20] and a rearrangement argument, where the solution to the even L^p Minkowski problem is applied to each level set of a function. A less circuitous proof also using the L^p Petty projection inequality, as well as the optimal L^p Sobolev norm and the CNV inequality (1.4), is presented in Section 7.

2 Preliminaries

Throughout this paper, $u\cdot x$ denotes the standard inner product of $u,x\in\mathbb{R}^n$, and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. For $1\le p<\infty$ and a measurable function $f:\mathbb{R}^n\to\mathbb{R}$, let $|f|_p$ denote the L^p norm of f and $L^p(\mathbb{R}^n)$ the corresponding space of L^p-bounded functions on $\mathbb{R}^n$.

An L^p_{loc} function $f:\mathbb{R}^n\to\mathbb{R}$ *has* L^p *weak derivative*, if there exists a measurable function $\nabla f:\mathbb{R}^n\to\mathbb{R}^n$ such that $|\nabla f|\in L^p(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} v(x)\cdot\nabla f(x)\,dx=-\int_{\mathbb{R}^n} f(x)\nabla\cdot v(x)\,dx, \tag{2.1}$$

for every compactly supported smooth vector field $v:\mathbb{R}^n\to\mathbb{R}^n$. The function ∇f is called the *weak gradient* of f, and the L^p norm of $|\nabla f|$ is denoted by $|\nabla f|_p$.

The norm dual to $\|\cdot\|$ is denoted by $\|\cdot\|_*$, where

$$\|u\|_* = \sup\left\{\frac{u\cdot x}{\|x\|} : x \in \mathbb{R}^n\backslash\{0\}\right\}, \tag{2.2}$$

for each $u \in \mathbb{R}^n$. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^p weak derivative, we denote

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} \left\|\nabla f(x)\right\|_*^p dx\right)^{1/p}. \tag{2.3}$$

We will call this an L^p *Sobolev norm of* f, even though it is only a seminorm.

Throughout this paper, a *convex body* is always assumed to be an origin-symmetric compact convex set in $\mathbb{R}^n$ with nonempty interior. A measure is always assumed to be a positive finite Borel measure.

The volume (i.e., Lebesgue measure) of a convex body K will be denoted by $V(K)$. A convex body K defines a norm $|\cdot|_K$ on $\mathbb{R}^n$ given by

$$|x|_K = \inf\left\{t > 0 : \frac{x}{t} \in K\right\} \tag{2.4}$$

for each $x \in \mathbb{R}^n$. The polar body K^* of K is defined by

$$K^* = \{u \in \mathbb{R}^n : u \cdot x \le 1 \text{ for each } x \in K\}. \tag{2.5}$$

Note that $|\cdot|_{K^*}$ is the norm dual to $|\cdot|_K$ and also the support function of K. The boundary of K will be denoted by ∂K.

The standard unit ball in $\mathbb{R}^n$ will be denoted by B and its volume by ω_n.

3 The L^p Minkowski problem

We begin by recalling basics that we need from the Brunn-Minkowski theory of convex bodies and its L^p extension (see Schneider [29] for details regarding the classical Brunn-Minkowski theory).

If K, L are convex bodies and $0 < t < \infty$, then the Minkowski combination $K + tL$ is defined by

$$|\cdot|_{(K+tL)^*} = |\cdot|_{K^*} + t|\cdot|_{L^*}. \tag{3.1}$$

As an aside, note that $K + tL = \{x + ty : x \in K,\ y \in L\}$.

The *mixed volume* $V_1(K, L)$ *of* K *and* L is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{t \to 0+} \frac{V(K + tL) - V(K)}{t}. \tag{3.2}$$

A fundamental fact is that corresponding to each convex body K is a unique Borel measure $S(K, \cdot)$ on the unit sphere S^{n-1} such that

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} |u|_{L^*} \, dS(K, u), \tag{3.3}$$

for each convex body L. The measure $S(K, \cdot)$ is called the *surface area measure* of K.

Let $h = |\cdot|_{K^*}$ denote the support function of K, and $h^* = |\cdot|_K$ the support function of K^*. Note that

$$h^*(x) = 1, \quad \text{for each } x \in \partial K. \tag{3.4}$$

Recall that the Gauss map assigns to each point of the boundary of a sufficiently smooth convex body in $\mathbb{R}^n$ its outer unit normal. Since h^* is a convex function (and therefore differentiable almost everywhere) and constant along the boundary of K, the Gauss map $\gamma : \partial K \to S^{n-1}$ can be defined almost everywhere on ∂K by

$$\gamma = \frac{\nabla h^*}{|\nabla h^*|}. \tag{3.5}$$

It follows from the definition of the dual norm that $h(\nabla h^*(x)) = 1$, for almost every $x \in \mathbb{R}^n$. This and the homogeneity (of degree 1) of h give

$$h\big(\gamma(x)\big) = \frac{1}{\big|\nabla h^*(x)\big|}. \tag{3.6}$$

Let $\sigma(\partial K, \cdot)$ be the $(n-1)$-dimensional volume measure induced on ∂K by the standard Euclidean structure on $\mathbb{R}^n$. It turns out that the surface area measure is given by

$$S(K, \cdot) = \gamma_* \sigma(\partial K, \cdot), \tag{3.7}$$

where γ_* denotes the pushforward induced by the Gauss map γ. If the boundary ∂K is strictly convex and smooth, then

$$S(K, \cdot) = \frac{du}{\kappa\big(\gamma^{-1}(u)\big)}, \tag{3.8}$$

where du is the standard Lebesgue measure on S^{n-1}, and $\kappa : \partial K \to \mathbb{R}$ is the Gauss curvature of the hypersurface ∂K.

Recall that a measure μ on the unit sphere S^{n-1} is said to be *even*, if it assumes the same values on antipodal Borel sets. The even Minkowski problem can be stated as follows. Given an even Borel measure μ on the unit sphere S^{n-1}, does there exist a convex body K whose surface area measure is μ? Or, equivalently, does there exist a convex body K such that

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} |u|_{L^*} \, d\mu(u), \tag{3.9}$$

for each convex body L?

Lutwak [17] showed how elements of the classical Brunn-Minkowski theory can be extended to a more general L^p Brunn-Minkowski theory (see, e.g., [5, 6, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 28, 30]) by using L^p Minkowski sums first introduced by Firey. The essential details are reviewed below.

Suppose $1 \le p < \infty$. If K, L are convex bodies, and $0 < t < \infty$, then the L^p *Minkowski-Firey combination* $K +_p tL$ is defined by

$$|\cdot|^p_{(K+_p tL)^*} = |\cdot|^p_{K^*} + t^p |\cdot|^p_{L^*}. \tag{3.10}$$

The L^p *mixed volume of* K *and* L is defined by

$$V_p(K, L) = \frac{p}{n} \lim_{t \to 0} \frac{V(K +_p t^{1/p} L) - V(K)}{t}, \tag{3.11}$$

and can be viewed as an L^p surface area of ∂K with respect to the geometric structure induced by the norm $|\cdot|_L$. It generalizes the Euclidean surface area of K, which is given by $nV_1(K, B)$, where B is the standard unit ball in $\mathbb{R}^n$. Note that

$$V_p(K, K) = V(K). \tag{3.12}$$

A fundamental inequality that we need is the following special case of the L^p *Minkowski inequality* [17].

Lemma 3.1. If $1 \le p < \infty$ and K, L are origin-symmetric convex bodies in $\mathbb{R}^n$, then

$$V_p(K, L) \ge V(K)^{1-p/n} V(L)^{p/n}. \tag{3.13}$$

Equality holds if and only if $L = tK$ for some $t > 0$. □

This inequality generalizes the classical isoperimetric inequality, where $p = 1$ and $L = B$. The following is a well-known and useful consequence.

Lemma 3.2. If K and L are convex bodies such that

$$\frac{V_p(K,Q)}{V(K)} = \frac{V_p(L,Q)}{V(L)} \tag{3.14}$$

for each convex body Q, then $K = L$. □

Proof. Setting $Q = K$ gives, by (3.12) and Lemma 3.1,

$$1 = \frac{V_p(K,K)}{V(K)} = \frac{V_p(L,K)}{V(L)} \geq \left(\frac{V(K)}{V(L)}\right)^{p/n}. \tag{3.15}$$

This gives $V(K) \leq V(L)$; setting $Q = L$ gives the reverse inequality. From the equality conditions of Lemma 3.1, L is a dilate of K. Since $V(K) = V(L)$, the bodies must be the same. ■

It was shown in [17] that corresponding to each convex body K is a unique Borel measure $S_p(K,\cdot)$ on the unit sphere S^{n-1} such that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} |u|_{L^*}^p \, dS_p(K,u), \tag{3.16}$$

for each convex body L. The measure $S_p(K,\cdot)$ is called the L^p *surface area measure* of K. One easily observes that for every $t > 0$,

$$S_p(tK,\cdot) = t^{n-p} S_p(K,\cdot). \tag{3.17}$$

It was also shown in [17] that the L^p surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to $S(K,\cdot) = S_1(K,\cdot)$, and that for the Radon-Nikodym derivative we have

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h^{1-p}, \tag{3.18}$$

where $h = |\cdot|_{K^*}$ is the support function of K.

The even L^p *Minkowski problem.* Given an even Borel measure μ on S^{n-1}, does there exist a convex body K such that $\mu = S_p(K,\cdot)$? Or, equivalently, does there exist a convex body K such that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} |u|_{L^*}^p \, d\mu(u), \tag{3.19}$$

for each convex body L?

Lutwak [17] gave an affirmative answer to this problem when $p \neq n$. The authors [23] introduced the *volume-normalized* L^p *Minkowski problem*, for which the case $p = n$ can be handled as well (The volume-normalized L^1 Minkowski problem was used earlier by Ball [3] to construct convex bodies with large shadow areas in all directions). See [5, 6, 11, 13] for recent progress on the L^p Minkowski problem when the given measure is not assumed to be even.

In particular, the authors solved the even case of the volume-normalized L^p Minkowski problem and proved the following in [23].

Theorem 3.3. If $1 \leq p < \infty$ and μ is an even Borel measure on the unit sphere S^{n-1}, then there exists a unique origin-symmetric convex body $\bar{K}$ such that

$$\frac{S_p(\bar{K},\cdot)}{V(\bar{K})} = \mu \tag{3.20}$$

if and only if the support of μ is not contained in any $(n-1)$-dimensional linear subspace. □

If $p \neq n$, Theorem 3.3 is equivalent to the solution to the even L^p Minkowski problem. Given a measure μ satisfying the assumptions of Theorem 3.3, then it follows from (3.17) that the unique solution to the even L^p Minkowski problem is obtained by letting

$$K = V(\bar{K})^{1/(p-n)}\bar{K}, \tag{3.21}$$

where $\bar{K}$ is the origin-symmetric convex body given by Theorem 3.3.

4 The functional L^p Minkowski problem

We begin by defining the L^p surface area measure of a Sobolev function.

Lemma 4.1. Given $1 \leq p < \infty$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^p weak derivative, there exists a unique finite Borel measure $S_p(f,\cdot)$ on S^{n-1} such that

$$\int_{\mathbb{R}^n} \varphi\big(-\nabla f(x)\big)^p \, dx = \int_{S^{n-1}} \varphi(u)^p \, dS_p(f,u), \tag{4.1}$$

for every nonnegative continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ homogeneous of degree 1. If f is not equal to a constant function almost everywhere, then the support of $S_p(f, \cdot)$ cannot be contained in any $(n-1)$-dimensional linear subspace. □

We call the measure $S_p(f, \cdot)$ given by the lemma above the L^p *surface area measure of the function* f.

Proof. Let $\Sigma = \{x : \nabla f(x) = 0\}$. Since

$$\psi \longmapsto \int_{\mathbb{R}^n \setminus \Sigma} \psi\left(- \frac{\nabla f(x)}{|\nabla f(x)|} \right) |\nabla f(x)|^p \, dx \tag{4.2}$$

defines a nonnegative bounded linear functional on the space of continuous functions on S^{n-1}, it follows by the Riesz representation theorem that there exists a unique Borel measure $S_p(f, \cdot)$ on S^{n-1} such that

$$\int_{\mathbb{R}^n \setminus \Sigma} \psi\left(- \frac{\nabla f(x)}{|\nabla f(x)|} \right) |\nabla f(x)|^p \, dx = \int_{S^{n-1}} \psi(u) \, dS_p(f, u), \tag{4.3}$$

for each continuous function $\psi : S^{n-1} \to \mathbb{R}$.

If $\varphi : \mathbb{R}^n \to [0, \infty)$ is continuous and homogeneous of degree 1, then $\varphi(-\nabla f(x)) = 0$, for each $x \in \Sigma$. This, the homogeneity of φ, and (4.3) with $\psi = \varphi^p$ (restricted to the unit sphere) give

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(-\nabla f(x))^p \, dx &= \int_{\mathbb{R}^n \setminus \Sigma} \varphi(-\nabla f(x))^p \, dx \\ &= \int_{\mathbb{R}^n \setminus \Sigma} \varphi\left(- \frac{\nabla f(x)}{|\nabla f(x)|} \right)^p |\nabla f(x)|^p \, dx \\ &= \int_{S^{n-1}} \varphi(u)^p \, dS_p(f, u). \end{aligned} \tag{4.4}$$

Thus, the measure $S_p(f, \cdot)$ satisfies (4.1) for each nonnegative continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ homogeneous of degree 1. The uniqueness of $S_p(f, \cdot)$ follows by observing that any measure $S_p(f, \cdot)$ on S^{n-1} that satisfies (4.1) defines the same linear functional as given by (4.2).

If the support of $S_p(f, \cdot)$ is contained in $H \cap S^{n-1}$, where, say, $H = \{x \in \mathbb{R}^n : x_n = 0\}$, then by (4.1),

$$\begin{aligned} 0 &= \int_{S^{n-1}} |u_n|^p \, dS_p(f, u) \\ &= \int_{\mathbb{R}^n} |\partial_n f(x)|^p \, dx. \end{aligned} \tag{4.5}$$

It follows that $\partial_n f = 0$ almost everywhere, and therefore, for each $i = 1, \dots, n$ and compactly supported smooth function χ,

$$\partial_n \partial_i (f * \chi) = 0, \tag{4.6}$$

where $f * \chi$ denotes the convolution of f and χ. Since $\partial_i f \in L^p(\mathbb{R}^n)$,

$$\partial_i (f * \chi) \in L^p\big(\mathbb{R}^n\big), \tag{4.7}$$

and since $f*\chi$ is smooth, it follows by (4.6), (4.7), and the mean value theorem that $\partial_i(f*\chi)$ is identically zero and $f*\chi$ is constant. This holds for every compactly supported smooth function χ, and therefore the function f must be constant almost everywhere. ∎

This leads naturally to the following.

The functional L^p *Minkowski problem.* Given a Borel measure μ on S^{n-1}, does there exist a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^p weak derivative such that $S_p(f, \cdot) = \mu$?

We answer this question for even measures by using the solution to the normalized even L^p Minkowski problem.

Theorem 4.2. If $1 \le p < \infty$ and μ is an even Borel measure on S^{n-1} whose support is not contained in any $(n-1)$-dimensional linear subspace, then there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^p weak derivative such that $S_p(f, \cdot) = \mu$. □

Proof. By Theorem 3.3, there exists an origin-symmetric convex body K such that

$$\frac{S_p(K, \cdot)}{V(K)} = \mu. \tag{4.8}$$

Let $\chi : [0, \infty) \to [0, \infty)$ be a smooth decreasing compactly supported function such that

$$\int_0^\infty \big(-\chi'(t)\big)^p t^{n-1}\, dt = \frac{1}{V(K)}, \tag{4.9}$$

and define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \chi\big(h^*(x)\big), \tag{4.10}$$

where, as before, $h^* = |\cdot|_K$ and $h = |\cdot|_{K^*}$. Since f is compactly supported, χ is smooth, and h^* is Lipschitz, it follows that the function f has weak L^p derivative.

Observe that the $(n-1)$-dimensional volume measure on $(h^*)^{-1}(t) = t\partial K$ induced by the standard Euclidean structure on $\mathbb{R}^n$ is given by

$$\sigma(t\partial K, t\omega) = t^{n-1}\sigma(\partial K, \omega), \tag{4.11}$$

for each $t > 0$ and Borel set $\omega \subset \partial K$. Therefore, by the coarea formula (see, e.g., Federer [9]) and the observation that ∇h^* is homogeneous of degree 0,

$$\begin{aligned}\int_{\mathbb{R}^n} F(x)\big|\nabla h^*(x)\big|^p dx &= \int_0^\infty \int_{(h^*)^{-1}(t)} F(x)\big|\nabla h^*(x)\big|^{p-1} d\sigma(t\partial K, x)dt \\ &= \int_0^\infty \int_{\partial K} F(tv)\big|\nabla h^*(v)\big|^{p-1} t^{n-1} d\sigma(\partial K, v)dt,\end{aligned} \tag{4.12}$$

for every $F \in L^1(\mathbb{R}^n)$.

Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be continuous and homogeneous of degree 1. By the chain rule, (4.12) and the homogeneity of φ, (3.4), (3.5), and (3.6), (4.9) and (3.7), and (3.18) and (4.8),

$$\begin{aligned}&\int_{\mathbb{R}^n} \varphi\big(-\nabla f(x)\big)^p dx \\ &\quad= \int_{\mathbb{R}^n} \varphi\big(-\chi'\big(h^*(x)\big)\nabla h^*(x)\big)^p dx \\ &\quad= \int_0^\infty \int_{\partial K} t^{n-1}\big(-\chi'(t)\big)^p \varphi\left(\frac{\nabla h^*(v)}{|\nabla h^*(v)|}\right)^p \big|\nabla h^*(v)\big|^{p-1} d\sigma(\partial K, v)dt \\ &\quad= \int_0^\infty \big(-\chi'(t)\big)^p t^{n-1} dt \int_{\partial K} \varphi\big(\gamma(v)\big)^p h\big(\gamma(v)\big)^{1-p} d\sigma(\partial K, v) \\ &\quad= \frac{1}{V(K)} \int_{S^{n-1}} \varphi(u)^p h(u)^{1-p} dS(K, u) \\ &\quad= \int_{S^{n-1}} \varphi(u)^p \frac{dS_p(K, u)}{V(K)} \\ &\quad= \int_{S^{n-1}} \varphi(u)^p d\mu(u).\end{aligned} \tag{4.13}$$

■

5 Existence and uniqueness of an optimal Sobolev norm

We use the even L^p Minkowski problem to show that for each function $f : \mathbb{R}^n \to \mathbb{R}$ with L^p weak derivative, there is a unique origin-symmetric convex body K whose L^p surface area measure is equal to the L^p surface area measure of f and that a dilate of this body is the unit ball for the optimal L^p Sobolev norm of f. This construction establishes a fundamental connection between functions on $\mathbb{R}^n$ and convex bodies in $\mathbb{R}^n$.

Theorem 5.1. If $1 \leq p < \infty$ and $f : \mathbb{R}^n \to \mathbb{R}$ has L^p weak derivative, then there exists a unique origin-symmetric convex body $K = K_p f$ such that

$$\int_{\mathbb{R}^n} \varphi\big(-\nabla f(x)\big)^p \, dx = \frac{1}{V(K)} \int_{S^{n-1}} \varphi(u)^p \, dS_p(K, u), \tag{5.1}$$

for every even continuous function $\varphi : \mathbb{R}^n \to [0, \infty)$ that is homogeneous of degree 1. □

A consequence of this theorem is that the L^p Sobolev norm of f is equal to a suitably normalized L^p mixed volume of the convex body $K_p f$ and the unit ball of the norm used to define the Sobolev norm. Specifically, if we set $\varphi = |\cdot|_{Q^*}$, then it follows by (5.1) and (3.16) that

$$\frac{1}{n} \int_{\mathbb{R}^n} |\nabla f(x)|_{Q^*}^p \, dx = \frac{V_p(K, Q)}{V(K)}, \tag{5.2}$$

for each origin-symmetric convex body Q.

A still elusive complete solution to the L^p Minkowski problem should provide necessary and sufficient conditions on a function f to guarantee the existence of a not necessarily origin-symmetric convex body K for which (5.1) holds for all continuous nonnegative functions φ that are homogeneous of degree 1 (but not necessarily even).

Proof. Let μ be the even part of the measure $S_p(f, \cdot)$ on S^{n-1}. By Theorem 3.3, there exists an origin-symmetric convex body K such that

$$\frac{S_p(K, \cdot)}{V(K)} = \mu. \tag{5.3}$$

If $\varphi : \mathbb{R}^n \to [0, \infty)$ is an even continuous function that is homogeneous of degree 1, then it follows by Lemma 4.1 and (4.8) that

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi\big(-\nabla f(x)\big)^p \, dx &= \int_{S^{n-1}} \varphi(u)^p \, dS_p(f, u) \\ &= \int_{S^{n-1}} \varphi(u)^p \, d\mu(u) \\ &= \frac{1}{V(K)} \int_{S^{n-1}} \varphi(u)^p \, dS_p(K, u). \end{aligned} \tag{5.4}$$

If K_1 and K_2 are both origin-symmetric convex bodies satisfying (5.1), then by (5.2) and Lemma 3.2, $K_1 = K_2$. ■

Corollary 5.2. Suppose $1 \leq p < \infty$. If $f : \mathbb{R}^n \to \mathbb{R}$ has L^p weak derivative, then there is, among all norms on $\mathbb{R}^n$ whose unit ball has the same volume as the Euclidean unit ball, a unique norm $\|\cdot\|$ that minimizes $\|\nabla f\|_p$. That norm is given by

$$\|\cdot\| = \left(\frac{V(K)}{\omega_n}\right)^{1/n} |\cdot|_K, \tag{5.5}$$

where $K = K_p f$. Moreover,

$$\|\nabla f\|_p = n^{1/p}\omega_n^{1/n} V(K_p f)^{-1/n}. \tag{5.6}$$

□

Proof. Note that (5.5) is equivalent to

$$\|\cdot\|_* = \left(\frac{\omega_n}{V(K)}\right)^{1/n} |\cdot|_{K^*}. \tag{5.7}$$

For each norm $|\cdot|_L$ such that $V(L) = \omega_n$, it follows by (5.2), the L^p Minkowski inequality (3.13), (3.12), (5.2) again, and (5.7) that

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\nabla f(x)\right|_{L^*}^p dx &= n\frac{V_p(K,L)}{V(K)} \\
&\geq nV(K)^{-p/n}\omega_n^{p/n} \\
&= n\left(\frac{\omega_n}{V(K)}\right)^{p/n}\frac{V_p(K,K)}{V(K)} \\
&= \left(\frac{\omega_n}{V(K)}\right)^{p/n}\int_{\mathbb{R}^n}\left|\nabla f(x)\right|_{K^*}^p dx \\
&= \int_{\mathbb{R}^n}\left\|\nabla f(x)\right\|_*^p dx.
\end{aligned} \tag{5.8}$$

Uniqueness of the norm $\|\cdot\|$ follows from the equality condition of the L^p Minkowski inequality (3.13).

Note that equation (5.6) is contained in the last four lines of (5.8). ■

Theorems 4.2 and 3.3 imply the following converse to Theorem 5.1.

Proposition 5.3. Suppose $1 \leq p < \infty$. If K is an origin-symmetric convex body, then there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ with weak L^p derivative such that $K_p f = K$. □

The convex body $K_p f$ encodes the geometry of the level sets of f. In particular, if all of the level sets are dilates of an origin-symmetric convex body K, then $K_p f$ is a dilate of K.

The following proposition describes how $K_p f$ behaves if f is composed with an invertible linear transformation.

Proposition 5.4. Suppose $1 \leq p < \infty$. If $f : \mathbb{R}^n \to \mathbb{R}$ has L^p weak derivative, and $\phi \in SL(n)$, then

$$K_p \left(f \circ \phi^{-1} \right) = \phi \left(K_p f \right). \tag{5.9}$$

□

Proof. Using the definitions of the L^p Minkowski-Firey combination and L^p mixed volume, it is straightforward to verify the well-known fact that for every pair of convex bodies K and L,

$$\frac{V_p(\phi K, L)}{V(\phi K)} = \frac{V_p\left(K, \phi^{-1} L\right)}{V(K)}. \tag{5.10}$$

Using the identity $|\phi^{-t} \cdot |_{L^*} = | \cdot |_{(\phi^{-1}L)^*}$, where ϕ^{-t} denotes the inverse transpose of ϕ, and making the change of variables $y = \phi(x)$ give

$$\int_{\mathbb{R}^n} \left|\nabla\left(f \circ \phi^{-1}\right)(y)\right|_{L^*}^p dy = \int_{\mathbb{R}^n} \left|\nabla f(x)\right|_{(\phi^{-1}L)^*}^p dx. \tag{5.11}$$

Let $K = K_p f$ and $K_\phi = K_p(f \circ \phi^{-1})$. By (3.16), (5.2), (5.11), (5.2) again, and (5.10),

$$\frac{V_p\left(K_\phi, L\right)}{V\left(K_\phi\right)} = \frac{V_p(\phi K, L)}{V(\phi K)}, \tag{5.12}$$

for each convex body L. The proposition now follows by Lemma 3.2. ■

It is easily verified that if $f : \mathbb{R}^n \to \mathbb{R}$ has weak L^p derivative and $g : \mathbb{R}^n \to \mathbb{R}$ is given by

$$g(x) = tf(cx + y), \tag{5.13}$$

for each $x \in \mathbb{R}^n$, where $t, c > 0$ and $y \in \mathbb{R}^n$, then $K_p g = t^{-1} c^{n/p-1} K_p f$. Combining this with Proposition 5.4 gives

$$K_p \left(tf \circ \Phi^{-1} \right) = t^{-1} |\phi|^{-1/p} \phi \left(K_p f \right), \tag{5.14}$$

for each $t > 0$ and invertible affine transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Phi(x) = \phi(x) + y, \tag{5.15}$$

where $y \in \mathbb{R}^n$, $\phi \in GL(n)$, and $|\phi|$ denotes the absolute value of the determinant of ϕ.

6 Sharp affine inequalities

Let $\mathrm{Aff}(n)$ denote the group of invertible affine transformations of $\mathbb{R}^n$. There is a natural left action of $\mathbb{R}\backslash\{0\} \times \mathrm{Aff}(n)$ on functions $f : \mathbb{R}^n \to \mathbb{R}$, given by $f \mapsto tf \circ \Phi^{-1}$, for each $(t, \Phi) \in \mathbb{R}\backslash\{0\} \times \mathrm{Aff}(n)$. An *affine inequality* for a class of functions $f : \mathbb{R}^n \to \mathbb{R}$ is an inequality $L[f] \leq R[f]$, where L and R are functionals such that

$$\frac{L[tf \circ \Phi]}{R[tf \circ \Phi]} = \frac{L[f]}{R[f]}, \tag{6.1}$$

for each $(t, \Phi) \in \mathbb{R}\backslash\{0\} \times \mathrm{Aff}(n)$. The inequality is *sharp* if there exists a function f for which equality holds, and such a function is called an *extremal* function for the inequality. If f is extremal, then so is $tf \circ \Phi$. In other words, the set of extremal functions is invariant under the left action of $\mathbb{R}\backslash\{0\} \times \mathrm{Aff}(n)$.

Corollary 5.2 can be used to establish sharp affine Sobolev inequalities. For example, it leads to a family of sharp affine inequalities, stated below in Theorem 6.1, that extend the Cordero-Nazaret-Villani inequalities.

For $x \in \mathbb{R}$, denote $x_+ = \max\{x, 0\}$. If $1 < p < n$, and $r \in (0, np/(n-p)]$, define $w : [0, \infty) \to [0, \infty)$ by

$$w(t) = \begin{cases} \left(1 + (r-p)t^{p/(p-1)}\right)_+^{p/(p-r)} & \text{if } r \neq p, \\ \exp\left(-pt^{p/(p-1)}\right) & \text{if } r = p. \end{cases} \tag{6.2}$$

For $p = 1$, let

$$w(t) = \begin{cases} 1 & \text{if } t \leq 1, \\ 0 & \text{if } t > 1. \end{cases} \tag{6.3}$$

Let

$$W(x) = w\big(|x|\big). \tag{6.4}$$

Let $q, \alpha, c_{p,r,n} \in \mathbb{R}$ satisfy

$$\begin{aligned} q &= \left(1 - \frac{1}{p}\right) r + 1, \\ \frac{1-\alpha}{q} + \frac{\alpha}{r} &= \frac{1}{p} - \frac{1}{n}, \\ c_{p,r,n} &= \frac{|\nabla W|_p}{|W|_q^{1-\alpha} |W|_r^{\alpha}}. \end{aligned} \tag{6.5}$$

By Corollary 5.2, there is a norm $\|\cdot\|$ such that the volume of $K_p f$ suitably normalized is equal to $\|\nabla f\|_p$. By this and the CNV inequalities (1.4), we get the following.

Theorem 6.1. Suppose $1 \le p < n$ and $0 < r \le np/(n-p)$. If $f \in L^q(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ has L^p weak derivative, then f satisfies the sharp affine inequality

$$V\big(K_p f\big)^{-1/n} \ge \tilde{c}_{p,r,n} |f|_q^{1-\alpha} |f|_r^{\alpha}, \tag{6.6}$$

where

$$\tilde{c}_{p,r,n} = n^{-1/p} \omega_n^{-1/n} c_{p,r,n}, \tag{6.7}$$

and q, α, and $c_{p,r,n}$ are given by (6.5). Equality holds if there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$, $a \in \mathbb{R}$, $\sigma \in (0,\infty)$, and $x_0 \in \mathbb{R}^n$, such that

$$f(x) = a w\left(\frac{\|x - x_0\|}{\sigma}\right), \tag{6.8}$$

for all $x \in \mathbb{R}^n$, where w is given by (6.2) if $p > 1$ and (6.3) if $p = 1$. □

That the sharp inequality (6.6) is affine follows from (5.14). Note that the set of extremal functions in Theorem 6.1 is infinite-dimensional, and therefore the group $\mathbb{R} \times \mathrm{Aff}(n)$ does not act transitively on this set. This is in contrast to Theorem 7.2, where the set of extremal functions is finite-dimensional, and the group $\mathbb{R} \times \mathrm{Aff}(n)$ acts transitively on that set.

7 The sharp affine L^p Gagliardo-Nirenberg inequalities

In this section, we show how the optimal Sobolev norm can be used to give a new straightforward proof of the sharp affine L^p Sobolev inequality proved by Zhang [34] for the case $p = 1$ and the authors [22] for the case $1 < p < n$. We begin by recalling the crucial geometric inequality underlying the analytic inequality, as well as some definitions needed to state the theorem.

Associated with an origin-symmetric convex body K is the convex body $\Gamma_{-p}K$, which is the unit ball of the norm on $\mathbb{R}^n$ given by

$$|x|_{\Gamma_{-p}K}^p = \frac{1}{V(K)} \int_{S^{n-1}} |u \cdot x|^p \, dS_p(K, u). \tag{7.1}$$

The body $\Gamma_{-p}K$ is called the *normalized* L^p *polar projection body of* K.

The range of the operator Γ_{-p} is the class of n-dimensional central slices of the L_p-ball. The integral transform used to define Γ_{-p} is the L^p-*cosine transform*, which is also the Fourier transform for even homogeneous functions of degree $-n-p$ on $\mathbb{R}^n$ (see Koldobsky [14] for details).

Petty established the case $p = 1$ (see, e.g., [10, 29, 32]) and the authors [20] established the case $p > 1$ of the following geometric inequality (see Campi and Gronchi [4] for a different approach).

Theorem 7.1 (L^p Petty projection inequality)**.** If $1 \leq p < \infty$ and K is a convex body, then

$$V\big(\Gamma_{-p}K\big) \leq a_{p,n} V(K), \tag{7.2}$$

where

$$a_{p,n} = \left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{p+n}{2}\right)}{2\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{p+1}{2}\right)}\right]^{n/p}. \tag{7.3}$$

Equality holds if and only if K is an ellipsoid. □

These concepts were extended from bodies to functions by Zhang [34] for $p = 1$ and the authors [22] for $p > 1$.

If $1 \leq p < \infty$ and $f : \mathbb{R}^n \to \mathbb{R}$ has L^p weak derivative, then the L^p *polar projection body* of f is defined to be the unit ball $B_p f$ of the norm on $\mathbb{R}^n$ given by

$$|x|_{B_p f} = \left(\int_{\mathbb{R}^n} \big|x \cdot \nabla f(y)\big|^p \, dy\right)^{1/p}. \tag{7.4}$$

We observe the volume of $B_p f$ can be given directly in terms of the function f by

$$V\big(B_p f\big) = \frac{1}{\Gamma\left(\frac{n}{p}+1\right)} \int_{\mathbb{R}^n} \exp\left(-\int_{\mathbb{R}^n} \big|x \cdot \nabla f(y)\big|^p \, dy\right) dx. \tag{7.5}$$

By (7.4), Theorem 5.1, and (7.1),

$$B_p f = \Gamma_{-p} K_p f. \tag{7.6}$$

In [20] it is shown that for each $\phi \in GL(n)$ and each convex body K, we have $\Gamma_{-p}\phi K = \phi\Gamma_{-p}K$. This and (5.14) give

$$B_p\left(tf \circ \Phi^{-1}\right) = t^{-1}|\phi|^{-1/p}\phi\big(B_p f\big), \tag{7.7}$$

for each $t > 0$ and invertible affine transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Phi(x) = \phi(x) + y, \tag{7.8}$$

where $y \in \mathbb{R}^n$ and $\phi \in GL(n)$.

The identity (7.6), the L^p Petty projection inequality (Theorem 7.1), and Corollary 5.2 lead to the following sharp affine L^p Gagliardo-Nirenberg inequalities. In contrast to Theorem 6.1, the extremal functions for this theorem are defined in terms of an inner product norm and have ellipsoids as level sets.

Theorem 7.2. Suppose $1 \leq p < n$ and $0 < r \leq np/(n-p)$. If $f \in L^q(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ has L^p weak derivative, then f satisfies the sharp affine inequality

$$V\big(B_p f\big)^{-1/n} \geq C_{p,r,n} |f|_q^{1-\alpha} |f|_r^{\alpha}, \tag{7.9}$$

where

$$C_{p,r,n} = n^{-1/p} \big(\omega_n a_{p,n}\big)^{-1/n} c_{p,r,n}, \tag{7.10}$$

q, α, and $c_{p,r,n}$ are given by (6.5), and $a_{p,n}$ is given by (7.3). Equality holds if there exists an inner product norm $\|\cdot\|$ on $\mathbb{R}^n$, $a \in \mathbb{R}$, $\sigma \in (0, \infty)$, and $x_0 \in \mathbb{R}^n$, such that

$$f(x) = a w\left(\frac{\|x - x_0\|}{\sigma}\right), \tag{7.11}$$

for all $x \in \mathbb{R}^n$, where w is given by (6.2) if $p > 1$ and (6.3) if $p = 1$. □

Proof. That the sharp inequality (7.9) is affine follows from (7.7).

By (7.6) and the L^p Petty projection inequality (Theorem 7.1), (5.6), and the CNV inequality (1.4),

$$\begin{aligned} V\big(B_p f\big)^{-1/n} &\geq a_{p,n}^{-1/n} V\big(K_p f\big)^{-1/n} \\ &= n^{-1/p} \big(\omega_n a_{p,n}\big)^{-1/n} \|\nabla f\|_p \\ &\geq C_{p,r,n} |f|_q^{1-\alpha} |f|_r^{\alpha}, \end{aligned} \tag{7.12}$$

where $\|\nabla f\|_p$ is the optimal L^p Sobolev norm of f. This proves (7.9).

The equality conditions for (7.9) follow from the equality conditions for the L^p Petty projection inequality (Theorem 7.1) and the affine CNV inequality (Theorem 6.1). ■

The case $r = pn/(n-p)$ of Theorem 7.2 is the sharp affine L^p Sobolev inequality established for $p = 1$ by Zhang [34] and for $1 < p < n$ by the authors [22].

8 Problem

Is there a direct solution to the functional even L^p Minkowski problem that does not make use of the solution to the even L^p Minkowski problem?

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