# LIE DERIVATIVE OF A DIFFERENTIAL FORM

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## 1. The flow of a vector field

Given a smooth $n$-manifold $Y$ and a smooth vector field $V$ on $Y$, let $F : [0, T) \times Y \to Y$, where $T > 0$, be the flow map of $V$, i.e., the unique map that satisfies, for each $(t, y) \in [0, T) \times Y$

$$F(0, y) = y \tag{1}$$

$$(F_* \partial_t)(t, y) = V(F(t, y)). \tag{2}$$

For each $t \in [0, T)$, denote $f_t = F(t, \cdot) : Y \to Y$.

## 2. Interior product

Given a $(k+1)$-form $\theta$ and a vector $v_0$, let $\langle v_0, \omega\rangle$ be the $k$-form such that for any vectors $v_1, \dots, v_k$,

$$\langle v_1 \otimes \cdots \otimes v_k, \langle v_0, \theta\rangle\rangle = \langle v_0 \otimes \cdots \otimes v_k, \theta\rangle$$

## 3. Orientation

Let $I = [0, t]$, where $t > 0$. An everywhere nonvanishing a differential $k$-form $\Theta$ on an $k$-manifold $Z$ with boundary determines an orientation on $Z$, and $dt \wedge \Theta$ an orientation on $I \times Z$. For each $z \in \partial(I \times Z)$, if $\eta \in T_x X$ is an outward pointing vector, then the orientation at $z$ of the boundary of $I \times Z$ is

$$\langle \eta, dt \wedge \Theta\rangle = \begin{cases} -\Theta & \text{on } \{0\} \times Z \\ \Theta & \text{on } \{t\} \times Z \\ -dt \wedge \langle \eta, \Theta\rangle & \text{on } I \times \partial X. \end{cases}$$

## 4. Integration

For each closed interval $I = [0, t]$, oriented $k$-manifold $Z$, and differential $(k + 1)$-form $\Theta$ on $I \times Z$,

$$\int_{I \times Z} \Theta = \int_{\tau=0}^{\tau=t} \left( \int_Z \langle \partial_\tau, \Theta\rangle \right) d\tau.$$

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## 5. Parameterized flow

Let $X$ be a smooth $m$-manifold with boundary. Given a smooth map $\phi_0 : X \to Y$, define $\Phi : [0,T) \times X \to Y$, where, for each $(t,x) \in [0,T) \times X$,

$$\Phi(t,x) = F(t, \phi_0(x)).$$

By (2), for each $(t,x) \in [0,T] \times X$,

$$(\Phi_* \partial_t)(t,x) = V(\Phi(t,x)).$$

For each $t \in [0,T)$, let $\phi_t = \Phi(t,\cdot) : X \to Y$. Observe that $\phi_t = f_t \circ \phi_0$ and therefore, by the chain rule, $(\phi_t)_* = (f_t)_* \circ (\phi_0)_*$.

## 6. The Lie derivative of a differential form

**Lemma 1.** *If $\omega$ is a smooth differential $m$-form on $Y$, then, for each $t \in [0,T)$,*

$$\int_X \phi_t^* \omega - \int_X \phi_0^* \omega = \int_{\tau=0}^{\tau=t} \left( \int_X \phi_\tau^* (\langle V, d\omega \rangle + d\langle V, \omega \rangle) \right) d\tau \tag{3}$$

*Proof.* Let $I = [0,t]$. On one hand,

$$\begin{aligned}
\int_{I \times X} d(\Phi^* \omega) &= \int_{\tau=0}^{\tau=t} \left( \int_X \langle \partial_\tau, d(\Phi^* \omega) \rangle \right) d\tau \\
&= \int_{\tau=0}^{\tau=t} \left( \int_X \langle \partial_\tau, d(\Phi^* \omega) \rangle \right) d\tau \\
&= \int_{\tau=0}^{\tau=t} \left( \int_X \Phi^* \langle \Phi_* \partial_\tau, d\omega \rangle \right) d\tau \\
&= \int_{\tau=0}^{\tau=t} \left( \int_X \Phi^* \langle V, d\omega \rangle \right) d\tau \\
&= \int_{\tau=0}^{\tau=t} \left( \int_X \phi_\tau^* \langle V, d\omega \rangle \right) d\tau
\end{aligned} \tag{4}$$

On the other hand, by Stokes's theorem twice,

$$\begin{aligned}
\int_{I\times X} d(\Phi^*\omega) &= \int_{\partial(I\times X)} \Phi^*\omega \\
&= \int_{\{t\}\times X} \Phi^*\omega - \int_{\{0\}\times X} \Phi^*\omega + \int_{I\times\partial X} \Phi^*\omega \\
&= \int_X \phi_t^*\omega - \int_X \phi_0^*\omega - \int_{\tau=0}^{\tau=t} \left( \int_{\partial X} \langle \partial_\tau, \Phi^*\omega \rangle \right) d\tau \\
&= \int_X \phi_t^*\omega - \int_X \phi_0^*\omega - \int_{\tau=0}^{\tau=t} \left( \int_{\partial X} \langle \partial_\tau, \phi_\tau^*\omega \rangle \right) d\tau \\
&= \int_X \phi_t^*\omega - \int_X \phi_0^*\omega - \int_{\tau=0}^{\tau=t} \left( \int_X d\langle \partial_\tau, \phi_\tau^*\omega \rangle \right) d\tau \\
&= \int_X \phi_t^*\omega - \int_X \phi_0^*\omega - \int_{\tau=0}^{\tau=t} \left( \int_X \phi_\tau^*(d\langle (\phi_\tau)_*\partial_\tau, \omega \rangle) \right) d\tau \\
&= \int_X \phi_t^*\omega - \int_X \phi_0^*\omega - \int_{\tau=0}^{\tau=t} \left( \int_X \phi_\tau^*(d\langle V, \omega \rangle) \right) d\tau
\end{aligned} \tag{5}$$

The lemma now follows by combining (4) and (5). □

**Corollary 2.** *If $\omega$ is a differential $m$-form on $Y$, then*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} f_t^*\omega = \langle V, d\omega \rangle + d\langle V, \omega \rangle. \tag{6}$$

*Proof.* Equation (3) is equivalent to

$$\int_X \phi_0^* f_t^*\omega - \int_X \phi_0\omega = \int_{\tau=0}^{\tau=t} \left( \int_X \phi_0^* f_\tau^*(\langle V, d\omega \rangle + d\langle V, \omega \rangle) \right) d\tau$$

Differentiating this with respect to $t$ and evaluating at $t = 0$ gives

$$\int_X \phi_0^* \left( \left.\frac{\partial}{\partial t}\right|_{t=0} f_t^*\omega \right) = \int_X \phi_0^* f_t^*(\langle V, d\omega \rangle + d\langle V, \omega \rangle).$$

Since this holds for any parameterization $\phi_0 : X \to Y$, (6) must hold. □