THE CRAMER–RAO INEQUALITY FOR STAR BODIES

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Abstract. Associated with each body K in Euclidean n-space $\mathbb{R}^n$ is an ellipsoid $\Gamma_2 K$ called the Legendre ellipsoid of K. It can be defined as the unique ellipsoid centered at the body's center of mass such that the ellipsoid's moment of inertia about any axis passing through the center of mass is the same as that of the body.

In an earlier paper the authors showed that corresponding to each convex body $K \subset \mathbb{R}^n$ is a new ellipsoid $\Gamma_{-2} K$ that is in some sense dual to the Legendre ellipsoid. The Legendre ellipsoid is an object of the dual Brunn–Minkowski theory, while the new ellipsoid $\Gamma_{-2} K$ is the corresponding object of the Brunn–Minkowski theory.

The present paper has two aims. The first is to show that the domain of Γ_{-2} can be extended to star-shaped sets. The second is to prove that the following relationship exists between the two ellipsoids: If K is a star shaped set, then

$$\Gamma_{-2} K \subset \Gamma_2 K,$$

with equality if and only if K is an ellipsoid centered at the origin. This inclusion is the geometric analogue of one of the basic inequalities of information theory – the Cramer-Rao inequality.

Associated with each body K in Euclidean n-space $\mathbb{R}^n$ is an ellipsoid $\Gamma_2 K$ called the Legendre ellipsoid of K. The Legendre ellipsoid is a basic concept from classical mechanics. It can be defined as the unique ellipsoid centered at the body's center of mass such that the ellipsoid's moment of inertia about any axis passing through the center of mass is the same as that of the body.

In [26] the authors showed that corresponding to each convex body $K \subset \mathbb{R}^n$ is a new ellipsoid $\Gamma_{-2} K$. The results in this paper hint at a remarkable duality between this new ellipsoid and the Legendre ellipsoid.

The present paper has two aims. The first is to show that the domain of Γ_{-2} can be extended to star-shaped sets. The second is to prove that the following relationship exists between the two ellipsoids: If K is a star shaped set, then

$$\Gamma_{-2} K \subset \Gamma_2 K, \tag{1}$$

with equality if and only if K is an ellipsoid centered at the origin. This inclusion is the geometric analogue of one of the basic inequalities of information theory – the Cramer-Rao inequality.

The Brunn-Minkowski theory (often called the theory of mixed volumes) is the heart of analytic convex geometry. Many of the fundamental ingredients of the theory were developed by Minkowski a century ago. Schneider's book [28] is the classical reference for the subject.

Over the years the tools of the Brunn-Minkowski theory have proven to be remarkably effective in solving inverse problems for which the data involves projections of convex bodies. A quarter of a century ago, the elements of a dual Brunn-Minkowski theory were introduced in [22] (and related papers). The basic idea of the dual theory is to replace the projections

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of the Brunn-Minkowski theory with intersections. In [23] it was shown that there is in fact a "dictionary" between the theories. Not only do concepts like the "elementary mixed volumes" of the classical theory become the "elementary dual mixed volumes" of the dual theory, but even objects such as the "projection bodies" of the Brunn-Minkowski theory have dual counterparts, "intersection bodies", in the dual theory. In fact, it was this dual notion of "intersection body" that played a key role in the ultimate solution of the Busemann-Petty problem (see e.g. Gardner [11, 12, 13], Zhang [29, 30], Koldobsky [17, 18, 19, 20], Gardner, Koldobsky, and Schlumprecht [15]). Gardner's book on geometric tomography [14] is an excellent reference for the interplay between the classical and dual theories.

Minkowski showed that what was to become known as the Brunn-Minkowski theory could be developed naturally by combining the notion of volume with an addition of convex bodies now known as Minkowski addition. In the early 1960's, Firey [9] introduced and studied an L_p generalization of Minkowski addition. In the 1990's, in [24, 25], these Minkowski-Firey L_p-sums were combined with the notion of volume to form embryonic L_p versions of the Brunn-Minkowski theory.

It is easily seen that the classical Legendre ellipsoid belongs to the dual Brunn-Minkowski theory. This observation led the authors to the obvious question: What is the dual analog of the Legendre ellipsoid in the Brunn-Minkowski theory? The answer was given by the new ellipsoid introduced in [26]. This new ellipsoid actually belongs to the L_2-Brunn-Minkowski theory.

The nature of the duality between the Brunn-Minkowski theory and the dual Brunn-Minkowski theory is not understood. The objects of study of the Brunn-Minkowski theory are convex bodies, while the objects of study of the dual Brunn-Minkowski theory are star bodies. The basic functionals of the Brunn-Minkowski theory are often expressed as integrals involving the support and curvature functions. The basic functionals of the dual Brunn-Minkowski theory are integrals involving radial functions. A first step in understanding the nature of the duality between the Brunn-Minkowski theory and its dual is to extend some of the functionals of the Brunn-Minkowski theory so that they are defined for star bodies (rather than convex bodies) and to provide new definitions of these functionals that involve only radial functions (rather than support and curvature functions). In this article, this first step is accomplished for one object of the Brunn-Minkowski theory: the Γ_{-2}-ellipsoid.

One of the central problems in information theory is how to extract useful information from noisy signals. Let $x_0 \in R^n$ be the transmitted signal. A simple model for the received signal is a random vector $x \in \mathbb{R}^n$ with a probability distribution $p(x - x_0)\,dx$ on $\mathbb{R}^n$, where the probability measure $p(x)\,dx$ has mean 0.

Suppose that the same signal is transmitted repeatedly and that $x_1, \ldots, x_N$ are the received signals. What is the best estimate for the transmitted signal, and what is the error of this estimate?

One possible estimate is the mean

$$\overline{x} = \frac{x_1 + \cdots + x_N}{N}.$$

By the central limit theorem, as N becomes large, the distribution of the random variable $\overline{x}$ approaches a Gaussian with mean x_0 and covariance matrix $C/\sqrt{N}$, where the matrix C is given by

$$C_{ij} = \int_{\mathbb{R}^n} x_i x_j p(x)\,dx.$$

Another estimate is the maximum likelihood estimate x_M, which is obtained by maximizing the log–likelihood function

$$L(x) = \sum_{i=1}^{N} \log p(x_i - x).$$

Fisher [10] and Doob [7] (also, see Appendix I of [1]) showed that as N becomes large, the distribution of the random variable x_M approaches a Gaussian with mean x_0 and covariance matrix $F^{-1}/\sqrt{N}$, where the matrix F is known as the Fisher information matrix and is given by

$$F_{ij} = \int_{\mathbb{R}^n} \frac{\partial(\log p)}{\partial x^i} \frac{\partial(\log p)}{\partial x^j} p\, dx.$$

A fundamental result in information theory is the Cramer-Rao inequality (see, for example, [5]), which states that the covariance and Fisher information matrices satisfy the inequality

$$v \cdot Cv \geq v \cdot F^{-1} v, \tag{2}$$

for all $v \in \mathbb{R}^n$. Moreover, equality holds for all $v \in \mathbb{R}^n$ if and only if the distribution p is Gaussian. The Cramer-Rao inequality is important, because it shows that the error in the maximum likelihood estimate is smaller than the mean estimate and that the mean estimate is optimal only if the distribution is Gaussian.

It has previously been observed [4, 5, 6] that there exists some connection between the subject of information theory and the Brunn-Minkowski theory. The authors believe that the true connection is between information theory and the L_2-Brunn-Minkowski theory. The authors have found small bits of an embryonic “dictionary” connecting the subject of information theory and the L_2-Brunn-Minkowski theory. In this dictionary a probability distribution corresponds to a convex body and the entropy power of the distribution to the volume of the body.

Associated with the covariance matrix C of a probability distribution is the ellipsoid

$$E_C = \{v \in \mathbb{R}^n \,:\, v \cdot C^{-1} v \leq 1\}.$$

In our dictionary this ellipsoid corresponds to the Legendre ellipsoid $\Gamma_2 K$ of a convex body K. Associated with the Fisher information matrix F is the ellipsoid

$$E_F = \{v \in \mathbb{R}^n \,:\, v \cdot Fv \leq 1\},$$

which corresponds to the ellipsoid $\Gamma_{-2} K$. The Cramer–Rao inequality (2) is equivalent to the statement that

$$E_F \subset E_C, \tag{3}$$

with equality holding if and only if the probability distribution is Gaussian. Using the dictionary, the main result of this paper, (1), corresponds to (3).

The definition of the Γ_{-2}–ellipsoid uses the derivative of the radial function of the star body and therefore would appear to require some differentiability assumptions for the star body. We show that, remarkably, no such assumptions are necessary for either the definition or the results of this paper.

1. Notation and overview

A bounded set $K \subset \mathbb{R}^n$ that is star–shaped about the origin is uniquely determined by its *radial function,* $\rho_K : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$, where

$$\rho_K(x) = \sup\{\lambda \geq 0 : \lambda x \in K\}.$$

A *star body* is a bounded set that is star-shaped about the origin and whose radial function is positive and continuous.

Throughout this paper the boundary of K will be denoted ∂K, and dy will denote the density associated with the $(n-1)$–dimensional Hausdorff measure on ∂K. The unit sphere in $\mathbb{R}^n$ will be denoted S^{n-1}, and the density associated with $(n-1)$–dimensional Hausdorff measure on S^{n-1} will be denoted du.

Given a convex body $K \subset \mathbb{R}^n$ containing the origin in its interior, the ellipsoid $\Gamma_{-2}K$ was defined in [26] as the body whose radial function is given by

$$\rho_{\Gamma_{-2}K}(x)^{-2} = \frac{1}{V(K)} \int_{\partial K} \frac{(x \cdot \nu(y))^2}{y \cdot \nu(y)}\, dy, \tag{4}$$

where $\cdot$ denotes the standard inner product on $\mathbb{R}^n$, $\nu(y) \in S^{n-1}$ is the outer unit normal at $y \in \partial K$. This formula can be used to define $\Gamma_{-2}K$ for any body $K \subset \mathbb{R}^n$ with sufficiently smooth boundary.

In §2 we use polar coordinates to obtain the following new formula for $\Gamma_{-2}K$:

Proposition 3. *Given a convex body $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n \backslash \{0\}$,*

$$\rho_{\Gamma_{-2}K}(x)^{-2} = \frac{1}{V(K)} \int_{S^{n-1}} (x \cdot \nabla \rho_K(u))^2 \rho_K(u)^{n-4}\, du.$$

In §§3–5 we present preliminary definitions and lemmas that are needed for the remainder of the paper.

We show in §6 how to define for a star body $K \subset \mathbb{R}^n$ the set $\Gamma_{-2}K \subset \mathbb{R}^n$. In §10 it will also be shown that this set is an ellipsoid that is possibly degenerate. If the boundary of K is Lipschitz, then $\Gamma_{-2}K$ is a nondegenerate ellipsoid. On the other hand, if the boundary of K is sufficiently singular, then $\Gamma_{-2}K$ is just a single point, namely the origin.

The following extension of Lemma 1* in [26] is proved in §7:

Lemma 13. *If $K \subset \mathbb{R}^n$ is a star body and $\phi \in \mathrm{GL}(n)$,*

$$\Gamma_{-2}\phi K = \phi \Gamma_{-2}K.$$

In §8 we derive a formula for the volume of K, and in §9 we use this formula to establish our main result:

Theorem. *If $K \subset \mathbb{R}^n$ is a star body, then*

$$\Gamma_{-2}K \subset \Gamma_2 K,$$

with equality holding if and only if K is an ellipsoid centered at the origin.

Recall that a set $E \subset \mathbb{R}^n$ is an ellipsoid centered at the origin, if there exists an $n \times n$ positive definite symmetric matrix A such that

$$E = \{x \,:\, x \cdot Ax \leq 1\}.$$

A set $E \subset \mathbb{R}^n$ is a degenerate ellipsoid centered at the origin if there exists a proper subspace $L \subset \mathbb{R}^n$ such that $E \subset L$ and E is an ellipsoid in L.

2. A new formula for the Γ_{-2}–ellipsoid

Given a convex body $K \subset \mathbb{R}^n$ containing the origin in its interior, there is a natural parameterization of the boundary ∂K given by the map $\phi_K : S^{n-1} \to \partial K$, where $S^{n-1} \subset \mathbb{R}^n$ is the unit sphere and

$$\phi_K(u) = \rho_K(u)u.$$

Moreover, the radial function ρ_K is Lipschitz and, for almost every $y \in \partial K$, there exists a unique outer unit normal $\nu(y)$ to ∂K at y (see, for example, page 53 of [28] for a proof of these facts).

When the meaning is clear, the subscript K in ρ_K and ϕ_K may be suppressed. To change the variable of integration in (4) from $y \in \partial K$ to $u \in S^{n-1}$, the following two lemmas are needed.

Lemma 1. *If $K \subset \mathbb{R}^n$ is a convex body containing the origin in its interior, then for almost every $u \in S^{n-1}$:*

$$\frac{\nu(\phi(u))}{\phi(u) \cdot \nu(\phi(u))} = -\frac{\nabla \rho(u)}{\rho(u)^2},$$

where $\nabla \rho$ denotes the gradient of ρ in $\mathbb{R}^n$.

Proof. Since ρ is constant along the boundary ∂K, its gradient is normal to the boundary. In other words, given $y \in \partial K$, there exists $\lambda(y) \in \mathbb{R}$ such that

$$\nabla \rho(y) = \lambda(y)\nu(y).$$

On the other hand, since ρ is homogeneous of degree -1,

$$\begin{aligned} 1 &= \rho(y) \\ &= -y \cdot \nabla \rho(y) \\ &= -\lambda(y) y \cdot \nu(y). \end{aligned}$$

Therefore,

$$\lambda(y) = -\frac{1}{y \cdot \nu(y)},$$

and

$$\nabla \rho(y) = -\frac{\nu(y)}{y \cdot \nu(y)}.$$

Substituting in $y = \phi(u)$ and observing that $\nabla \rho$ is homogeneous of degree -2 yields the desired formula. $\square$

The formula for surface area measure of ∂K in polar coordinates is given in the following:

Lemma 2. *If $K \subset \mathbb{R}^n$ is a convex body containing the origin in its interior, then*

$$dy = \frac{\rho_K(u)^n}{\phi(u) \cdot \nu(\phi(u))}\, du.$$

The parameterization $\phi : S^{n-1} \to \partial K$ is Lipschitz and therefore differentiable almost everywhere. By Theorem 3.2.3 in [8], the change of measure is given by the determinant of the Jacobian. A straightforward computation yields the formula above.

Proposition 3. *Given a convex body $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n \backslash \{0\}$,*

$$\rho_{\Gamma_{-2}K}(x)^{-2} = \frac{1}{V(K)} \int_{S^{n-1}} (x \cdot \nabla\rho_K(u))^2 \rho_K(u)^{n-4}\, du, \tag{5}$$

where $\nabla\rho_K$ is the gradient of ρ_K in $\mathbb{R}^n$.

Proof. By Lemmas 1 and 2,

$$\begin{aligned}
\rho_{\Gamma_{-2}K}(x)^{-2} &= \frac{1}{V(K)} \int_{\partial K} \frac{(x \cdot \nu(y))^2}{y \cdot \nu(y)}\, dy \\
&= \frac{1}{V(K)} \int_{S^{n-1}} \frac{(x \cdot \nu(\phi(u)))^2}{(\phi(u) \cdot \nu(\phi(u)))^2} \rho_K(u)^n\, du \\
&= \frac{1}{V(K)} \int_{S^{n-1}} (x \cdot \nabla\rho_K(u))^2 \rho_K(u)^{n-4}\, du.
\end{aligned}$$

□

The operator Γ_{-2} can be extended immediately to the class of star bodies with Lipschitz radial function using either formula (4) or (5). To extend it to the entire class of star bodies requires a little more work. The next three sections contain preliminary definitions and results that are needed to this end.

3. A projection of the sphere onto the cylinder

The projection of a vector $x \in \mathbb{R}^n$ into each tangent space of S^{n-1} defines a vector field,

$$\hat{x}(u) = x - (x \cdot u)u. \tag{6}$$

We want to use the integral curves of $\hat{x}$ to define coordinates on S^{n-1}. This can be done as follows.

Given $x \in \mathbb{R}^n \backslash \{0\}$, let $x^\perp = \{v \in \mathbb{R}^n : v \cdot x = 0\}$ and let

$$\langle x \rangle = \frac{x}{|x|}.$$

There is a natural projection of S^{n-1} onto the cylinder $\mathbb{R}x + (S^{n-1} \cap x^\perp)$ obtained by mapping the unit vector u to the intersection of the cylinder with the ray containing u. The inverse of this projection is given by

$$\begin{aligned}
M_x : \mathbb{R} \times (S^{n-1} \cap x^\perp) &\to S^{n-1} \backslash \{\langle -x \rangle, \langle x \rangle\} \\
(t, \phi) &\mapsto u = \frac{\phi + \langle x \rangle \sinh |x|t}{\cosh |x|t}.
\end{aligned}$$

A straightforward calculation shows that

$$x \cdot u = |x| \tanh |x|t \tag{7}$$

and

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{x - \phi|x| \sinh |x|t}{(\cosh |x|t)^2} \\
&= x - (x \cdot u)u.
\end{aligned}$$

This implies that for each $f \in C^1(S^{n-1})$,

$$\nabla_{\hat{x}(M_x(t,\phi))} f(M_x(t,\phi)) = \frac{\partial (f \circ M_x)}{\partial t}(t,\phi), \tag{8}$$

where $\nabla_v f$ denotes the directional derivative of f in the direction v. Also,

$$du = \frac{|x|\,dt\,d\phi}{(\cosh|x|t)^{n-1}}, \tag{9}$$

where $d\phi$ is the density corresponding to the $(n-2)$–dimensional Hausdorff measure on $S^{n-1} \cap x^\perp$.

4. Differentiability along a vector field

In this section we make precise the notions of continuous differentiability and L_2 differentiability along the vector field $\hat{x}$ of a function on the unit sphere, without assuming any regularity in other directions.

Given $x \in \mathbb{R}^n$, let $C^0(S^{n-1} \cap x^\perp, C^1(\mathbb{R}))$ denote the space of continuous functions

$$f : \mathbb{R} \times (S^{n-1} \cap x^\perp) \to \mathbb{R}$$

such that $f(\cdot,\phi) \in C^1(\mathbb{R})$, for each $\phi \in S^{n-1} \cap x^\perp$. Define

$$C^1_x(S^{n-1}) = \{f \in C^0(S^{n-1}) \,:\, f \circ M_x \in C^0(S^{n-1} \cap x^\perp, C^1(\mathbb{R}))\}.$$

The restriction of any $f \in C^1_x(S^{n-1})$ to an integral curve of the vector field $\hat{x}$ is C^1. Therefore, $\nabla_{\hat{x}} f$ is well defined and given explicitly by

$$\nabla_{\hat{x}} f(M_x(t,\phi)) = \frac{\partial}{\partial t}[f(M_x(t,\phi))]. \tag{10}$$

The following chain rule holds:

Lemma 4. *Given a C^1 function ϕ and $f \in C^1_x(S^{n-1})$, the function $\phi \circ f \in C^1_x(S^{n-1})$. Moreover,*

$$\nabla_{\hat{x}}(\phi \circ f) = (\phi' \circ f)\nabla_{\hat{x}} f.$$

To define the notion of L_2 differentiability, we need the following integration by parts formulas:

Lemma 5. *Given $x \in \mathbb{R}^n$ and $f, g \in C^1_x(S^{n-1})$,*

$$\int_{S^{n-1}} f \nabla_{\hat{x}} g\,du = \int_{S^{n-1}} g[-\nabla_{\hat{x}} f + (n-1)(x \cdot u) f]\,du, \tag{11}$$

and

$$\int_{S^{n-1}} f \nabla_{\hat{x}} g\,du = -\int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} (g \circ M_x) \frac{\partial}{\partial t}\left[\frac{|x|(f \circ M_x)}{(\cosh|x|t)^{n-1}}\right] dt\,d\phi. \tag{12}$$

Proof. By (9) and (10),

$$\int_{S^{n-1}} f \nabla_{\hat{x}} g\,du = \int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} (f \circ M_x) \frac{\partial (g \circ M_x)}{\partial t}\,\frac{|x|\,dt\,d\phi}{(\cosh|x|t)^{n-1}}.$$

Equation (12) now follows by integrating by parts. By (9), (8), and (7),

$$\begin{aligned}\int_{S^{n-1}} f\nabla_{\hat{x}} g\, du &= -\int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} (g\circ M_x)\frac{\partial}{\partial t}\left[\frac{|x|(f\circ M_x)}{(\cosh|x|t)^{n-1}}\right] dt\, d\phi\\ &= -\int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} (g\circ M_x)\left[\frac{\partial(f\circ M_x)}{\partial t} - (n-1)|x|\tanh|x|t\right]\frac{|x|\,dt\,d\phi}{(\cosh|x|t)^{n-1}}\\ &= \int_{S^{n-1}} g[-\nabla_{\hat{x}} f + (n-1)(x\cdot u)f]\, du.\end{aligned}$$

□

Equation (11) can be used to define for any $g \in C^0(S^{n-1})$ the directional derivative $\nabla_{\hat{x}} g$ as a distribution. This motivates the following:

Definition. Given $x \in \mathbb{R}^n$ and $g \in C^0(S^{n-1})$, we say that $\nabla_{\hat{x}} g \in L_2(S^{n-1})$ if there exists a function $g' \in L_2(S^{n-1})$ such that for any $f \in C^1_x(S^{n-1})$,

$$\int_{S^{n-1}} g[-\nabla_{\hat{x}} f + (n-1)(x\cdot u)f]\, du = \int_{S^{n-1}} g' f\, du. \tag{13}$$

If such a function $g' \in L_2(S^{n-1})$ exists, it will be denoted $\nabla_{\hat{x}} g$.

Since $L_2(S^{n-1})$ is its own dual, we have

Lemma 6. *If $x \in \mathbb{R}^n$ and $g \in C^0(S^{n-1})$, then $\nabla_{\hat{x}} g \in L_2(S^{n-1})$ if and only if there exists $c > 0$ such that*

$$\int_{S^{n-1}} g[-\nabla_{\hat{x}} f + (n-1)(x\cdot u)f]\, du \le c\|f\|_2, \tag{14}$$

for each $f \in C^1_x(S^{n-1})$.

Lemma 5 now gives

Lemma 7. *If $g \in C^1_x(S^{n-1})$, the pointwise definition* (10) *of $\nabla_{\hat{x}} g$ agrees with the distributional definition* (13)*.*

5. Smoothing a function along a vector field

Fix $x \in \mathbb{R}^n\backslash\{0\}$. We want to define a smoothing operator that smooths only along the integral curves of the vector field $\hat{x}$.

Given $\tau > 0$, let χ_τ be a smooth nonnegative function on $\mathbb{R}$ that is supported in the interval $(-\tau, \tau)$ and satisfies

$$\int_{-\infty}^{\infty} \chi_\tau(t)\, dt = 1.$$

Using the map M_x defined in §3 we define for each $g \in L_2(S^{n-1})$ and $\tau > 0$, the function $g_\tau \in C^0(S^{n-1})$ by

$$g_\tau(M_x(t,\phi)) = \int_{-\infty}^{\infty} g(M_x(t-s,\phi))\chi_\tau(s)\, ds. \tag{15}$$

Observe that

$$g_\tau(M_x(t,\phi)) = \int_{-\infty}^{\infty} g(M_x(s,\phi))\chi_\tau(t-s)\, ds, \tag{16}$$

and g_τ extends continuously across $\langle x\rangle$ and $\langle -x\rangle \in S^{n-1}$.

Before stating and proving the main proposition about g_τ, we begin with two lemmas. The first is an analogue of the standard (and trivial) fact that convolution on $\mathbb{R}^n$ commutes with partial differentiation.

Lemma 8. *Given $g \in C^0(S^{n-1})$ and $x \in \mathbb{R}^n$, if $\nabla_{\hat{x}} g \in L_2(S^{n-1})$, then*

$$\nabla_{\hat{x}} g_\tau = (\nabla_{\hat{x}} g)_\tau. \tag{17}$$

Proof. It suffices to prove that

$$\frac{\partial(g_\tau \circ M_x)}{\partial t} = (\nabla_{\hat{x}} g)_\tau \circ M_x.$$

By (16) and (10), given any $f \in C^0(S^{n-1} \cap x^\perp)$,

$$\begin{aligned}
\int_{S^{n-1}\cap x^\perp} f(\phi) \frac{\partial(g_\tau \circ M_x)}{\partial t}(t,\phi)\, d\phi &= \int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} f(\phi) g(M_x(s,\phi)) \chi_\tau'(t-s)\, ds\, d\phi \\
&= -\int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} g(M_x(s,\phi)) \frac{\partial}{\partial s}[\chi_\tau(t-s) f(\phi)]\, ds\, d\phi \\
&= \int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} f(\phi)\chi_\tau(t-s)(\nabla_{\hat{x}(u)} g)(M_x(s,\phi))\, ds\, d\phi \\
&= \int_{S^{n-1}\cap x^\perp} f(\phi)(\nabla_{\hat{x}(u)} g)_\tau (M_x(t,\phi))\, d\phi.
\end{aligned}$$

This holds for each $f \in C^0(S^{n-1} \cap x^\perp)$ and thus yields (17). ∎

The next lemma is a weak form of Young's inequality.

Lemma 9. *There exists $T > 0$ such that for each $h \in L_2(S^{n-1})$ and $0 < \tau < T$,*

$$\|h_\tau\|_2 < 2\|h\|_2.$$

Proof. Observe that

$$\frac{\cosh |x|(t-s)}{\cosh |x| t} \le e^{|sx|}$$

for every $t \in \mathbb{R}$. Therefore, if

$$0 < \tau < \frac{\log 4}{(n-1)|x|},$$

it follows by Minkowski's inequality (see, for example, Theorem 2.4 in [21]) that

$$\begin{aligned}
\|h_\tau\|_2 &= \left[\int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} \left(\int_{-\infty}^{\infty} h(M_x(t-s,\phi))\chi_\tau(s)\, ds\right)^2 \frac{|x|\, dt\, d\phi}{(\cosh|x|t)^{n-1}}\right]^{1/2} \\
&\le \int_{-\infty}^{\infty} \chi_\tau(s) \left(\int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} h(M_x(t-s,\phi))^2 \frac{|x|\, dt\, d\phi}{(\cosh|x|t)^{n-1}}\right)^{1/2} ds \\
&= \int_{-\infty}^{\infty} \chi_\tau(s) \left(\int_{\mathbb{R}\times(S^{n-1}\cap x^\perp)} h(M_x(t-s,\phi))^2 \left(\frac{\cosh|x|(t-s)}{\cosh|x|t}\right)^{n-1} \frac{|x|\, dt\, d\phi}{(\cosh|x|(t-s))^{n-1}}\right)^{1/2} ds \\
&\le 2\|h\|_2.
\end{aligned}$$

∎

The main result of this section is the following:

Proposition 10. *Given $x \in \mathbb{R}^n \setminus \{0\}$ and $g \in C^0(S^{n-1})$ such that $\nabla_{\hat{x}} g \in L_2(S^{n-1})$, there exists a 1–parameter family of functions $g_\tau \in C^1_x(S^{n-1})$, for $\tau > 0$, such that the following hold:*

$$\lim_{\tau \to 0} \sup_{u \in S^{n-1}} |g_\tau(u) - g(u)| = 0, \tag{18}$$

and

$$\lim_{\tau \to 0} \|\nabla_{\hat{x}} g_\tau - \nabla_{\hat{x}} g\|_2 = 0. \tag{19}$$

Proof. Since g is uniformly continuous on S^{n-1}, there exist for each $\epsilon > 0$, quantities $\delta > 0$ and $T > 0$ such that

$$\begin{aligned} |g(M_x(t_1, \phi)) - g(M_x(t_2, \phi))| &< \epsilon; \\ |g(M_x(t, \phi)) - g(\langle x \rangle)| &< \frac{\epsilon}{2}; \\ |g(M_x(-t, \phi)) - g(\langle -x \rangle)| &< \frac{\epsilon}{2}; \end{aligned}$$

for every $\phi \in S^{n-1} \cap x^\perp$, $t \in (T - \delta, \infty)$, and $t_1, t_2 \in [-T, T]$ satisfying $|t_1 - t_2| < \delta$. It follows that given any $\tau \in (0, \delta/2)$,

$$\begin{aligned} |g_\tau(M_x(t, \phi)) - g(M_x(t, \phi))| &\le \int_{-\infty}^{\infty} |g(M_x(s, \phi)) - g(M_x(t, \phi))| \chi_\tau(t - s)\, ds \\ &< \epsilon, \end{aligned}$$

for every $(t, \phi) \in \mathbb{R} \times (S^{n-1} \cap x^\perp)$. This proves (18).

Given $\epsilon > 0$, there exists a function $f \in C^0(S^{n-1})$ such that $\|f - \nabla_{\hat{x}} g\|_2 < \epsilon$. By (18), f_τ converges uniformly to f. Therefore, there exists $T > 0$ such that $\|f_\tau - f\|_2 < \epsilon$, for any $\tau \in (0, T)$. By Lemmas 8 and 9, for any $\tau \in (0, T)$,

$$\begin{aligned} \|\nabla_{\hat{x}} g_\tau - \nabla_{\hat{x}} g\|_2 &\le \|\nabla_{\hat{x}} g_\tau - f_\tau\|_2 + \|f_\tau - f\|_2 + \|f - \nabla_{\hat{x}} g\|_2 \\ &= \|(\nabla_{\hat{x}} g - f)_\tau\|_2 + \|f_\tau - f\|_2 + \|f - \nabla_{\hat{x}} g\|_2 \\ &\le 2\|\nabla_{\hat{x}} g - f\|_2 + \|f_\tau - f\|_2 + \|f - \nabla_{\hat{x}} g\|_2 \\ &\le 4\epsilon. \end{aligned}$$

This proves (19). ∎

We can now prove the following analogue of Lemma 4:

Lemma 11. *Suppose $x \in \mathbb{R}^n$, ϕ is a C^1 function, and $g \in C^0(S^{n-1})$. If $\nabla_{\hat{x}} g \in L_2(S^{n-1})$, then $\nabla_{\hat{x}}(\phi \circ g) \in L_2(S^{n-1})$ and*

$$\nabla_{\hat{x}}(\phi \circ g) = (\phi' \circ g) \nabla_{\hat{x}} g.$$

Proof. By (18), as $\tau \to 0$, $\phi'\circ g_\tau$ converges uniformly to $\phi'\circ g$ and $\phi\circ g_\tau$ to $\phi\circ g$. Therefore, $(\phi'\circ g_\tau)\nabla_{\hat{x}} g_\tau$ converges in L_2 to $(\phi'\circ g)\nabla_{\hat{x}} g$, and, given any $f \in C^1_x(S^{n-1})$,

$$\begin{aligned}\int_{S^{n-1}} (\phi\circ g)[-\nabla_{\hat{x}} f + (n-1)(x\cdot u)f]\,du &= \lim_{\tau\to 0}\int_{S^{n-1}} (\phi\circ g_\tau)[-\nabla_{\hat{x}} f + (n-1)(x\cdot u)f]\,du \\ &= \lim_{\tau\to 0}\int_{S^{n-1}} [(\phi'\circ g_\tau)\nabla_{\hat{x}} g_\tau] f\,du \\ &= \int_{S^{n-1}} [(\phi'\circ g)\nabla_{\hat{x}} g] f\,du.\end{aligned}$$

The lemma now follows by (13). □

6. The Γ_{-2}–ellipsoid for star bodies

Given a star body $K \subset \mathbb{R}^n$, define $g_K : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ to be

$$g_K(x) = \begin{cases} \dfrac{\rho_K(x)^{\frac{n}{2}-1}}{\frac{n}{2}-1}, & \text{if } n > 2; \\ \log \rho_K(x), & \text{if } n = 2. \end{cases} \tag{20}$$

If $\rho_K : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ is C^1, then formula (5) can be rewritten as

$$\rho_{\Gamma_{-2}K}(x)^{-2} = \frac{1}{V(K)}\int_{S^{n-1}} (\nabla_x g_K(u))^2\,du, \tag{21}$$

where $\nabla_x g_K$ denotes the directional derivative of g_K (viewed as a function on $\mathbb{R}^n\backslash\{0\}$) in the direction u. In general, g_K is known only to be C^0, and its directional derivative does not necessarily exist. Nevertheless, it is still possible to define $\Gamma_{-2}K$.

Obviously, given $p \in \mathbb{R}$, any function $g \in C^1(S^{n-1})$ has a unique extension to $\mathbb{R}^n\backslash\{0\}$ that is homogeneous of degree p. Euler's equation says that for each $y \in \mathbb{R}^n\backslash\{0\}$,

$$\nabla_y g(y) = p g(y).$$

Therefore, given $x \in \mathbb{R}^n$,

$$\nabla_x g(u) = \nabla_{\hat{x}(u)} g(u) + p(x\cdot u)g(u), \tag{22}$$

for any $u \in S^{n-1}$. If we assume only that $g \in C^0(S^{n-1})$ and $\nabla_{\hat{x}} g \in L_2(S^{n-1})$, then the function $\nabla_x g \in L_2(S^{n-1})$ can be defined using (22).

In particular, if $g_K \in C^1(S^{n-1})$, then the homogeneity of ρ_K and Euler's equation imply that, for any $u \in S^{n-1}$,

$$\nabla_x g_K(u) = \nabla_{\hat{x}} g_K(u) - (x\cdot u)\rho_K(u)^{\frac{n}{2}-1}. \tag{23}$$

If we assume only that $\nabla_{\hat{x}} g_K \in L_2(S^{n-1})$, then equation (23) can be used to define $\nabla_x g_K \in L_2(S^{n-1})$. Lemma 11 ensures that (23) is consistent with the definition of $\nabla_x \rho_K$, as given by (22).

We can now define the set $\Gamma_{-2}K$.

Definition. Given a star body $K \subset \mathbb{R}^n$, define

$$\Gamma_{-2}K = \{x \in \mathbb{R}^n \,:\, \nabla_x g_K \in L_2(S^{n-1}) \text{ and } \|\nabla_x g_K\|_2^2 \le V(K)\}. \tag{24}$$

If the body K is convex, then this definition agrees with formula (5) and therefore with the original definition given in [26].

The following integration by parts formulas will be needed later:

Lemma 12. *Suppose $K \subset \mathbb{R}^n$ is a star body, and $x \in \mathbb{R}^n$ such that $\nabla_{\hat{x}} g_K \in L_2(S^{n-1})$. Then for every $f \in C^1_x(S^{n-1})$,*

$$\int_{S^{n-1}} f \nabla_x g_K \, du = \begin{cases} -\displaystyle\int_{S^{n-1}} g_K \nabla_x f \, du, & \text{if } n > 2 \\ -\displaystyle\int_{S^{n-1}} (\nabla_x f)(g_K - \log|f|) \, du, & \text{if } n = 2, \end{cases} \tag{25}$$

where f has been extended to be homogeneous of degree $-\frac{n}{2}$ on $\mathbb{R}^n \backslash \{0\}$.

Proof. First, assume that $g_K \in C^1_x(S^{n-1})$. The case $n > 2$ follows directly from equation (22), Lemma 4, and Lemma 5.

The case $n = 2$ requires the additional observations that $f \nabla_x(\log|f|) \in L_\infty(S^{n-1})$, and that

$$f(u)\nabla_x(\log|f(u)|) = \nabla_x f(u),$$

for almost every $u \in S^{n-1}$.

If $\nabla_{\hat{x}} g_K \in L_2(S^{n-1})$, then equation (25) holds for $(g_K)_\tau$, as defined by Lemma 10. The lemma now follows by taking the limit $\tau \to 0$. □

7. Invariance under GL(n)

Lemma 13. *If $K \subset \mathbb{R}^n$ is a star body and $\phi \in \mathrm{GL}(n)$,*

$$\Gamma_{-2}\phi K = \phi \Gamma_{-2} K.$$

Proof. Given a star body $K \subset \mathbb{R}^n$, let g_K be as defined by (20). Given a star body $L \subset \mathbb{R}^n$, let

$$f_L = \rho_L^{\frac{n}{2}}.$$

Recall that $\langle x \rangle = x/|x|$, for any $x \in \mathbb{R}^n$. Let $\phi \in \mathrm{SL}(n)$ and

$$v = \langle \phi^{-1} u \rangle.$$

Using the fact that

$$\rho_{\phi K}(u) = \rho_K(\phi^{-1}u),$$

and (25), we have for $n > 2$,

$$\begin{aligned} \int_{S^{n-1}} f_L \nabla_x g_{\phi K} \, du &= -\int_{S^{n-1}} g_{\phi K}(u) \nabla_x f_L(u) \, du \\ &= -\int_{S^{n-1}} g_K(\phi^{-1}u) \nabla_x f_L(u) \, du \\ &= -\int_{S^{n-1}} |\phi^{-1}u|^{1-\frac{n}{2}} g_K\left(\langle \phi^{-1}u \rangle\right) \nabla_x f_L\left(\langle \phi v \rangle\right) \, du \\ &= -\int_{S^{n-1}} g_K(v) \nabla_x f_L(\phi v) |\phi^{-1}u|^{-n} \, du \\ &= -\int_{S^{n-1}} g_K(v) \nabla_{\phi^{-1}x} f_{\phi^{-1}L}(v) \, dv. \end{aligned}$$

If $n = 2$, then

$$
\begin{aligned}
\int_{S^{n-1}} f_L \nabla_x g_{\phi K}\, du &= \int_{S^{n-1}} \rho_L \nabla_x \left(\log \frac{\rho_{\phi K}}{\rho_L} \right) du \\
&= \int_{S^{n-1}} -\nabla_x \rho_L(u) \log \frac{\rho_{\phi K}(u)}{\rho_L(u)}\, du \\
&= \int_{S^{n-1}} -\nabla_x \rho_L\left(\langle \phi v \rangle\right) \log \frac{\rho_K(\phi^{-1}u)}{\rho_{\phi^{-1}L}(\phi^{-1}u)}\, du \\
&= \int_{S^{n-1}} -\nabla_x \rho_L(\phi v) \log \frac{\rho_K(v)}{\rho_{\phi^{-1}L}(v)} |\phi^{-1}u|^{-2}\, du \\
&= \int_{S^{n-1}} -\nabla_{\phi^{-1}x} \rho_{\phi^{-1}L}(v) \log \frac{\rho_K(v)}{\rho_{\phi^{-1}L}(v)}\, dv \\
&= \int_{S^{n-1}} \rho_{\phi^{-1}L}(v) \nabla_{\phi^{-1}x} \left(\log \frac{\rho_K(v)}{\rho_{\phi^{-1}L}(v)} \right) dv \\
&= \int_{S^{n-1}} f_{\phi^{-1}L} \nabla_{\phi^{-1}x} g_K\, dv.
\end{aligned}
$$

Thus, for each dimension $n \geq 2$ and star body L with a C^1 radial function,

$$
\int_{S^{n-1}} \nabla_x g_{\phi K} f_L\, du = \int_{S^{n-1}} \nabla_{\phi^{-1}x} g_K f_{\phi^{-1}L}\, du.
$$

It follows by linearity that for any $f \in C^1(S^{n-1})$,

$$
\int_{S^{n-1}} \nabla_x g_{\phi K} f\, du = \int_{S^{n-1}} \nabla_{\phi^{-1}x} g_K f{\circ}\phi\, du.
$$

Therefore,

$$
\nabla_x g_{\phi K} \in L_2(S^{n-1}) \text{ if and only if } \nabla_{\phi^{-1}x} g_K \in L_2(S^{n-1}).
$$

Given any $f \in C^1(S^{n-1})$, extend it to $\mathbb{R}^n \backslash \{0\}$ as a function homogeneous of degree $-\frac{n}{2}$. There exists a star–shaped set L such that

$$
\rho_L^{\frac{n}{2}} = |f|.
$$

Observe that for any $\phi \in \mathrm{SL}(n)$,

$$
\rho_{\phi^{-1}L}^{\frac{n}{2}} = |f{\circ}\phi|,
$$

and therefore

$$
\begin{aligned}
\|f{\circ}\phi\|_2^2 &= nV(\phi^{-1}L) \\
&= nV(L) \\
&= \|f\|_2^2.
\end{aligned}
$$

It follows that

$$\begin{aligned}\|\nabla_x g_{\phi K}\|_2 &= \sup\left\{\int_{S^{n-1}} \nabla_x g_{\phi K} f\,du \,:\, \|f\|_2 = 1\right\}\\ &= \sup\left\{\int_{S^{n-1}} \nabla_{\phi^{-1}x} g_K f\circ\phi\,du \,:\, \|f\circ\phi\|_2 = 1\right\}\\ &= \|\nabla_{\phi^{-1}x} g_K\|_2,\end{aligned}$$

and

$$\begin{aligned}\Gamma_{-2}\phi K &= \{x \in \mathbb{R}^n \,:\, \nabla_x g_{\phi K} \in L_2(S^{n-1}) \text{ and } \|\nabla_x g_{\phi K}\|_2^2 \le V(\phi K)\}\\ &= \{x \in \mathbb{R}^n \,:\, \nabla_{\phi^{-1}x} g_K \in L_2(S^{n-1}) \text{ and } \|\nabla_{\phi^{-1}x} g_K\|_2^2 \le V(K)\}\\ &= \{\phi w \in \mathbb{R}^n \,:\, \nabla_w g_K \in L_2(S^{n-1}) \text{ and } \|\nabla_w g_K\|_2^2 \le V(K)\}\\ &= \phi\Gamma_{-2}K.\end{aligned}$$

For the case of dilation, let $\lambda > 0$. It is easily seen that

$$\nabla_x g_{\lambda K} = \nabla_{\lambda^{\frac{n}{2}-1}x} g_K = \lambda^{\frac{n}{2}} \nabla_{\lambda^{-1}x} g_K.$$

Therefore,

$$\begin{aligned}\Gamma_{-2}\lambda K &= \{x \in \mathbb{R}^n \,:\, \nabla_x g_{\lambda K} \in L_2(S^{n-1}) \text{ and } \|\nabla_x g_{\lambda K}\|_2^2 \le V(\lambda K)\}\\ &= \{x \in \mathbb{R}^n \,:\, \nabla_{\lambda^{-1}x} g_K \in L_2(S^{n-1}) \text{ and } \|\nabla_{\lambda^{-1}x} g_K\|_2^2 \le V(K)\}\\ &= \{\lambda w \in \mathbb{R}^n \,:\, \nabla_w g_K \in L_2(S^{n-1}) \text{ and } \|\nabla_w g_K\|_2^2 \le V(K)\}\\ &= \lambda\Gamma_{-2}K.\end{aligned}$$

□

8. A formula for volume

The following formula for the volume of a star body K is needed to prove that the Legendre ellipsoid contains the Γ_{-2}–ellipsoid:

Lemma 14. *Given $x \in S^{n-1}$ and a star body $K \subset \mathbb{R}^n$, if $\nabla_{\hat{x}} g_K \in L_2(S^{n-1})$, then*

$$V(K) = \int_{S^{n-1}} (x\cdot u)\rho_K(u)^{\frac{n}{2}+1}(-\nabla_x g_K(u))\,du.$$

Proof. For convenience we shall omit the subscript K, denoting ρ_K by ρ and g_K by g.

By (23), Lemma 11, and (22), the following holds in $L_2(S^{n-1})$:

$$\begin{aligned}\rho^{\frac{n}{2}+1}\nabla_x g &= \rho^{\frac{n}{2}+1}[\nabla_{\hat{x}} g - (x\cdot u)\rho^{\frac{n}{2}-1}]\\ &= \rho^{n-1}\nabla_{\hat{x}}\rho - (x\cdot u)\rho^n\\ &= \frac{1}{n}\nabla_{\hat{x}}(\rho^n) - (x\cdot u)\rho^n.\end{aligned}$$

Since $\nabla_{\hat{x}}(\rho^n) \in L_2(S^{n-1})$, it follows by (13) that

$$\begin{aligned}\int_{S^{n-1}} (x \cdot u)\rho^{\frac{n}{2}+1}(-\nabla_x g)\, du &= \int_{S^{n-1}} (x \cdot u) \left[-\frac{1}{n}\nabla_{\hat{x}}(\rho^n) + (x \cdot u)\rho^n \right] du \\ &= \int_{S^{n-1}} \frac{1}{n}[\nabla_{\hat{x}}(x \cdot u) + (x \cdot u)^2]\rho^n\, du \\ &= \frac{1}{n}\int_{S^{n-1}} \rho^n\, du \\ &= V(K).\end{aligned}$$

□

9. Inclusion and equality

Theorem. *If $K \subset \mathbb{R}^n$ is a star body, then*

$$\Gamma_{-2}K \subset \Gamma_2 K,$$

with equality holding if and only if K is an ellipsoid centered at the origin.

Proof. By applying a linear transformation to K, we can assume that its Legendre ellipsoid $\Gamma_2 K$ is the unit ball. This means that

$$\frac{1}{V(K)}\int_{S^{n-1}} (x \cdot u)^2 \rho_K(u)^{n+2}\, du = |x|^2 \tag{26}$$

for every $x \in \mathbb{R}^n$.

Let $x \in \Gamma_{-2}K$. First, it follows from the definition of $\Gamma_{-2}K$ that $\nabla_{\hat{x}} g_K \in L_2(S^{n-1})$. By Lemma 14, the Hölder inequality, (26), and (24),

$$\begin{aligned}|x|^2 &= \frac{1}{V(K)}\int_{S^{n-1}} (x \cdot u)\rho(u)^{\frac{n}{2}+1}(-\nabla_x g_K(u))\, du \\ &\le \left(\frac{1}{V(K)}\int_{S^{n-1}} (x \cdot u)^2\rho_K(u)^{n+2}\, du\right)^{1/2} \left(\frac{1}{V(K)}\int_{S^{n-1}} (\nabla_x g_K(u))^2\, du\right)^{1/2} \\ &\le |x|.\end{aligned} \tag{27}$$

Thus, $|x| \le 1$, and therefore, $\Gamma_{-2}K$ is contained in the unit ball. This proves that

$$\Gamma_{-2}K \subset \Gamma_2 K.$$

Suppose $\Gamma_{-2}K = \Gamma_2 K$. Since we are assuming that $\Gamma_2 K$ is the unit ball, so is $\Gamma_{-2}K$. This implies that for each $x \in S^{n-1}$, we have $\nabla_x g_K \in L_2(S^{n-1})$, and equality holds in all of the inequalities in (27). From the equality conditions of the Hölder inequality, there exists for each $x \in S^{n-1}$ a constant $c(x)$ such that

$$(x \cdot u)\rho_K(u)^{\frac{n}{2}+1} = c(x)\nabla_x g_K(u), \tag{28}$$

for almost every $u \in S^{n-1}$. Integrating the square of both sides with respect to $u \in S^{n-1}$ and using (24) and (26) shows that $c(x) = \pm 1$. Since (28) holds for all $x \in S^{n-1}$, it follows by (13) that for each $f \in C^1(\mathbb{R}^n\backslash\{0\})$ homogeneous of degree $-\frac{n}{2}-1$, if $n > 2$,

$$\begin{aligned}
\int_{S^{n-1}} \rho_K^{\frac{n}{2}+1} f\,du &= \sum_{i=1}^n \int_{S^{n-1}} (u\cdot e_i)(e_i\cdot u)\rho_K(u)^{\frac{n}{2}+1} f(u)\,du \\
&= \pm \sum_{i=1}^n \int_{S^{n-1}} (u\cdot e_i) f(u) \nabla_{e_i} g_K(u)\,du \\
&= \mp \sum_{i=1}^n \int_{S^{n-1}} \nabla_{e_i}((u\cdot e_i)f) g_K\,du \\
&= \mp \sum_{i=1}^n \int_{S^{n-1}} [(u\cdot e_i)\nabla_{e_i} f + f] g_K\,du \\
&= \mp \int_{S^{n-1}} (\nabla_u f + nf) g_K\,du \\
&= \mp \int_{S^{n-1}} f\rho^{\frac{n}{2}-1}\,du.
\end{aligned}$$

If $n = 2$, then for each $f \in C^1(\mathbb{R}^n\backslash\{0\})$ homogeneous of degree $-\frac{n}{2}-1$,

$$\begin{aligned}
\int_{S^{n-1}} \rho_K^{\frac{n}{2}+1} f\,du &= \sum_{i=1}^n \int_{S^{n-1}} (u\cdot e_i)(e_i\cdot u)\rho_K(u)^{\frac{n}{2}+1} f(u)\,du \\
&= \pm \sum_{i=1}^n \int_{S^{n-1}} (u\cdot e_i) f(u) \nabla_{e_i} g_K(u)\,du \\
&= \pm \sum_{i=1}^n \int_{S^{n-1}} [(u\cdot e_i) f \nabla_{e_i}(g_K - \log|f|^{1/2}) + \frac{1}{2}(u\cdot e_i)\nabla_{e_i} f]\,du \\
&= \mp \sum_{i=1}^n \int_{S^{n-1}} [(g - \log|f|^{1/2})\nabla_{e_i}[(u\cdot e_i)f] + \frac{1}{2} f \nabla_{e_i}(u\cdot e_i)]\,du \\
&= \mp \int_{S^{n-1}} f\,du.
\end{aligned}$$

Therefore

$$\rho^{\frac{n}{2}+1} = \pm\rho^{\frac{n}{2}-1},$$

in each dimension $n \geq 2$. Since ρ_K is positive and continuous, it must be the constant function 1. In other words, K is the unit ball centered at the origin. □

10. The set $\Gamma_{-2}K$ is an ellipsoid

Corollary. *If $K \subset \mathbb{R}^n$ is a star body, then the set $\Gamma_{-2}K \subset \mathbb{R}^n$ is an ellipsoid that is possibly degenerate.*

Proof. Suppose K is a star body and g_K is as defined by (20). Using (13), it is easily seen that for all $x, v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, if $\nabla_x g_K, \nabla_v g_K \in L_2(S^{n-1})$, then $\nabla_{\lambda x} g_K, \nabla_{x+v} g_K \in L_2(S^{n-1})$. It follows that the set

$$L = \{x \in \mathbb{R}^n \,:\, \nabla_x g_K \in L_2(S^{n-1})\}$$

is a linear subspace of $\mathbb{R}^n$. Moreover, $\Gamma_{-2}K \subset L$.

Let $\dim L = m \le n$. Choose a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ such that $e_1, \dots, e_m$ is a basis of L. Given $x \in L$, write

$$x = \sum_{i=1}^{m} x_i e_i.$$

Then $x \in \Gamma_{-2}K$ if and only if $x \cdot Ax \le V(K)$, where A is the nonnegative symmetric matrix whose (i, j)–th entry is

$$A_{ij} = \int_{S^{n-1}} (\nabla_{e_i} g_K)(\nabla_{e_j} g_K)\, du,$$

where $1 \le i, j \le m$. Observe that since $e_i, e_j \in L$, it follows from the Hölder inequality that the integral is bounded.

Now suppose A has a zero eigenvalue. Then the set $\Gamma_{-2}K$ is unbounded. However, the Legendre ellipsoid $\Gamma_2 K$ is bounded and, by the theorem, $\Gamma_{-2}K \subset \Gamma_2 K$. Therefore, $\Gamma_{-2}K$ is bounded. The contradiction shows that A is positive definite, and $\Gamma_{-2}K$ is a nondegenerate ellipsoid in L. □

References

[1] J. M. Borwein, A. S. Lewis, and D. Noll, *Maximum entropy reconstruction using derivative information. I. Fisher information and convex duality*, Math. Oper. Res. **21** (1996), no. 2, 442–468. MR **97e:**90114

[2] H. J. Brascamp and E. H. Lieb, *Best constants in Young's inequality, its converse, and its generalization to more than three functions*, Advances in Math. **20** (1976), no. 2, 151–173.

[3] ______, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis **22** (1976), no. 4, 366–389.

[4] M. H. M. Costa and T. M. Cover, *On the similarity of the entropy power inequality and the Brunn–Minkowski inequality*, IEEE Trans. Inform. Theory **30** (1984), 837–839.

[5] T. M. Cover and J. A. Thomas, *Elements of information theory*, Wily–Interscience, 1991.

[6] A. Dembo, T. M. Cover, and J. A. Thomas, *Information-theoretic inequalities*, IEEE Trans. Inform. Theory **37** (1991), no. 6, 1501–1518.

[7] J. L. Doob, *Probability and statistics*, Transactions of the AMS **36** (1934), 759–775.

[8] H. Federer, *Geometric measure theory*, Springer-Verlag New York Inc., New York, 1969, Die Grundlehren der mathematischen Wissenschaften, Band 153.

[9] Wm. J. Firey, *p–means of convex bodies*, Math. Scand. **10** (1962), 17–24.

[10] R. A. Fisher, *Theory of statistical estimation*, Philos. Trans. Roy. Soc. London Ser A **222** (1930), 309–368.

[11] R. J. Gardner, *On the Busemann-Petty problem concerning central sections of centrally symmetric convex bodies*, Bull. Amer. Math. Soc. (N.S.) **30** (1994), no. 2, 222–226.

[12] ______, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. of Math. (2) **140** (1994), no. 2, 435–447.

[13] ______, *Geometric tomography*, Notices Amer. Math. Soc. **42** (1995), 422–429.

[14] ______, *Geometric Tomography*, Cambridge University Press, Cambridge, 1995.

[15] R. J. Gardner, A. Koldobsky, and T. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math. (2) **149** (1999), no. 2, 691–703.

[16] O. Guleryz, E. Lutwak, D. Yang, and G. Zhang, *Information theoretic inequalities for contoured distributions*, preprint.

[17] Alexander Koldobsky, *Intersection bodies and the Busemann-Petty problem*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 11, 1181–1186.

[18] ______, *Intersection bodies in* $\mathbf{R}^4$, Adv. Math. **136** (1998), no. 1, 1–14.

[19] ______, *Intersection bodies, positive definite distributions, and the Busemann-Petty problem*, Amer. J. Math. **120** (1998), no. 4, 827–840.

[20] ______, *Second derivative test for intersection bodies*, Adv. Math. **136** (1998), no. 1, 15–25.

[21] E. H. Lieb and M. Loss, *Analysis*, American Mathematical Society, Providence, RI, 1997.
[22] E. Lutwak, *Dual mixed volumes*, Pacific J. Math. **58** (1975), no. 2, 531–538.
[23] ______, *Intersection bodies and dual mixed volumes*, Advances in Math. **71** (1988), 232–261.
[24] ______, *The Brunn–Minkowski–Firey theory I: mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.
[25] ______, *The Brunn–Minkowski–Firey theory II: affine and geominimal surface area*, Advances in Math. **118** (1996), 244–294.
[26] E. Lutwak, D. Yang, and G. Zhang, *A new ellipsoid associated with convex bodies*, Duke Mathematical Journal, **104** (2000), 375-390.
[27] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geometric Aspects of Functional Analysis (1987–88), Springer, Berlin, 1989, pp. 64–104.
[28] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge, 1993.
[29] G. Zhang, *Centered bodies and dual mixed volumes*, Trans. Amer. Math. Soc. **345** (1994), no. 2, 777–801.
[30] ______, *A positive solution to the Busemann-Petty problem in* $\mathbf{R}^4$, Ann. of Math. (2) **149** (1999), no. 2, 535–543.

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