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ON THE L_p-MINKOWSKI PROBLEM

ERWIN LUTWAK, DEANE YANG, AND GAOYONG ZHANG

ABSTRACT. A volume-normalized formulation of the L_p-Minkowski problem is presented. This formulation has the advantage that a solution is possible for all $p \geq 1$, including the degenerate case where the index p is equal to the dimension of the ambient space. A new approach to the L_p-Minkowski problem is presented, which solves the volume-normalized formulation for even data and all $p \geq 1$.

The *Minkowski problem* deals with existence, uniqueness, regularity, and stability of closed convex hypersurfaces whose Gauss curvature (as a function of the outer normals) is preassigned. Major contributions to this problem were made by Minkowski [M$_1$], [M$_2$], Aleksandrov [A$_2$], [A$_3$], [A$_4$], Fenchel and Jessen [FJ], Lewy [Le$_1$] [Le$_2$], Nirenberg [N], Calabi [Cal], Pogorelov [P$_1$], [P$_2$], Cheng and Yau [ChY], Caffarelli, Nirenberg, and Spruck [CNS], and others.

Variants of the Minkowski problem were presented by Gluck [Gl$_1$] and Singer [Si]. The survey of Gluck [Gl$_2$] still serves as an excellent introduction to the problem.

In this article we consider a generalization of the Minkowski problem known as the L_p*-Minkowski problem.* This generalization was studied in [Lu$_1$] and [LuO]. See Stancu [St$_1$], [St$_2$] and Umanskiy [U] for other recent work on the L_p-Minkowski problem.

In [Lu$_1$] a solution to the *even* L_p-Minkowski problem in $\mathbb{R}^n$ was given for all $p \geq 1$ (the case $p = 1$ is classical), except for $p = n$. The solution to the even L_p-Minkowski problem was one of the critical ingredients needed to obtain the sharp affine L_p Sobolev inequality [LuYZ$_1$].

The lack of a solution for the case $p = n$ is troubling. In this article we present a new *volume normalized* form of the classical Minkowski problem. This problem has a natural L_p analog that can (and will) be solved for all $p \geq 1$ for the *even data* case. It must be emphasized that, except for the critical case $p = n$, both the L_p-Minkowski problem and the *volume normalized* L_p-Minkowski problem are equivalent in that a solution to one will quickly and trivially provide a solution to the other. The road to the solution given here to the even volume normalized L_p-Minkowski problem is quite different from the path taken in [Lu$_1$] in solving the even L_p-Minkowski problem. The solution to the volume-normalized even L_p-Minkowski problem for all $p \geq 1$ is needed in [LuYZ$_2$].

A compact convex subset of Euclidean n-space $\mathbb{R}^n$ will be called a *convex body.* Associated with a convex body K is its *support function* $h(K, \cdot) : S^{n-1} \to \mathbb{R}$ which, for $u \in S^{n-1}$, is defined by $h(K, u) = \max\{u \cdot x : x \in K\}$. For each

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$u \in S^{n-1}$, the subset of K of the form $\{x \in K : x \cdot u = h(K,u)\}$ is called a *face* of K with outer unit normal u. If the face has positive *area* (i.e., $(n-1)$-dimensional volume), then it is called a *proper face* of K. The distance from the origin to the plane containing a proper face is called the *support number* associated with the face. If $u_1, \dots, u_N \in S^{n-1}$ and $c_1, \dots, c_N > 0$, then a convex body P of the form

$$P = \bigcap_{i=1}^{N} \{x \in \mathbb{R}^n : x \cdot u_i \le c_i\}$$

is called a *convex polytope*.

The *Minkowski problem with discrete data* asks: Under what conditions on the unit vectors $u_1, \dots, u_N$ and real numbers $a_1, \dots, a_N > 0$ does there exist a convex polytope with N proper faces whose outer unit normals are $u_1, \dots, u_N$ and such that the face with outer unit normal u_i has area a_i? Minkowski's solution to the problem is as follows:

If the unit vectors $u_1, \dots, u_N$ do not lie in a great subsphere of S^{n-1} and the positive numbers $a_1, \dots, a_N$ are such that

$$\sum_{i=0}^{N} a_i u_i = 0,$$

then there exists a convex polytope in $\mathbb{R}^n$ with N proper faces whose outer unit normals are $u_1, \dots, u_N$ and such that the face with outer unit normal u_i has area a_i. Furthermore, this polytope is unique, up to translation.

A special case is the solution of the *Minkowski problem with even discrete data*: If $u_1, \dots, u_N \in S^{n-1}$ do not lie in a great subsphere of S^{n-1} and $a_1, \dots, a_N > 0$ are given, then there exists a convex polytope in $\mathbb{R}^n$, symmetric about the origin, with $2N$ proper faces whose outer unit normals are $\pm u_1, \dots, \pm u_N$ such that the faces with outer unit normal $\pm u_i$ have area a_i. Furthermore, this polytope is unique (up to translation).

The *L_p-Minkowski problem with discrete data* asks the following question:

Suppose $\alpha \in \mathbb{R}$ is fixed. Under what conditions on N unit vectors $u_1, \dots, u_N$ and positive real numbers $a_1, \dots, a_N$ does there exist a convex polytope with N proper faces whose outer unit normals are $u_1, \dots, u_N$, and such that if f_i and h_i are the area and support number of the face with outer unit normal u_i, then

$$h_i^{\alpha} f_i = a_i, \qquad \text{for all } i.$$

Obviously, for the case $\alpha = 0$ the L_p-Minkowski problem reduces to the classical Minkowski problem.

A solution to the *L_p-Minkowski problem with discrete even data* was given in [Lu$_1$], as follows:

Suppose $\alpha \le 0$ and $\alpha \ne 1-n$. If the unit vectors $u_1, \dots, u_N$ do not lie in a great subsphere of S^{n-1} and $a_1, \dots, a_N > 0$ are given, then there exists a convex polytope in $\mathbb{R}^n$ that is symmetric about the origin, with $2N$ proper faces such that if f_i and h_i are the area and support numbers of the faces with outer unit normals $\pm u_i$, then

$$h_i^{\alpha} f_i = a_i, \qquad \text{for all } i.$$

Furthermore, the polytope is unique if $\alpha < 0$.

There is a Minkowski problem for arbitrary convex bodies. To state this problem, some preliminary terminology and notation is helpful.

A point x on the boundary ∂K is said to have an outer unit normal u if $x \cdot u = h(K, u)$; i.e., the point x has an outer normal u if x belongs to the face of K that has an outer normal u. (Obviously, a point of ∂K may have more than one outer unit normal.) The *surface area measure*, $S(K, \cdot\,)$, of a convex body K is a Borel measure on S^{n-1} that can be defined as follows: If ω is a Borel subset of S^{n-1}, then $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of points on ∂K that have an outer unit normal that is a member of the set ω.

If P is a polytope with N proper faces with areas $f_1, \ldots, f_N$ and corresponding normals $u_1, \ldots, u_N$, then the measure $S(P, \cdot\,)$ is a discrete measure whose support is $\{u_1, \ldots, u_N\}$ and such that

$$S(P, \{u_i\}) = f_i, \qquad \text{for all } i.$$

If K is a convex body whose boundary is sufficiently smooth and has positive Gauss curvature, then the Radon-Nikodym derivative of $S(K, \cdot\,)$, with respect to spherical Lebesgue measure, is a function on S^{n-1} whose value at the point $u \in S^{n-1}$ is the reciprocal Gauss curvature of ∂K at the point whose outer unit normal is u.

The Minkowski problem asks: Under what conditions on a measure μ on S^{n-1} does there exist a convex body K such that

$$S(K, \cdot\,) = \mu?$$

The answer for this problem is as follows:

If μ is a Borel measure on S^{n-1} whose support is not contained in a great subsphere of S^{n-1} and whose centroid is at the origin, i.e.,

$$\int_{S^{n-1}} u \, d\mu(u) = 0,$$

then there exists a convex body K such that

$$S(K, \cdot\,) = \mu.$$

Furthermore, the body K is unique, up to translation. For arbitrary convex bodies this solution is due to Aleksandrov [A$_2$], and Fenchel & Jessen [FJ].

To state the Minkowski problem with even data, recall that a measure is said to be *even* if it assumes the same values on antipodal Borel sets. The solution to the Minkowski problem with even data follows immediately from the general solution, and has the following simple formulation:

If μ is an even Borel measure on S^{n-1} whose support is not contained in a great subsphere of S^{n-1}, then there exists a convex body K, symmetric about the origin, such that

$$S(K, \cdot\,) = \mu.$$

Furthermore, the body K is unique (up to translation).

The L_p-Minkowski problem asks the following question:

Suppose $\alpha \in \mathbb{R}$ is fixed. Under what conditions on a measure μ on S^{n-1} does there exists a convex body K such that

$$h(K, \cdot\,)^{\alpha} dS(K, \cdot\,) = d\mu?$$

Obviously, for the case $\alpha = 0$ the L_p-Minkowski problem reduces to the classical Minkowski problem. For sufficiently smooth bodies and $\alpha = 1$ the problem was posed by Firey [Fi].

A partial solution to the L_p-Minkowski problem with even data was given in [Lu$_1$]:

Suppose $\alpha \le 0$ and $\alpha \ne 1-n$. If μ is an even Borel measure on S^{n-1} whose support is not contained in a great subsphere of S^{n-1}, then there exists a convex body K, symmetric about the origin, such that

$$h(K,\,\cdot\,)^\alpha dS(K,\,\cdot\,) = d\mu.$$

Furthermore, the body K is unique if $\alpha < 0$.

The restriction in the solution to the even L_p-Minkowski problem that $\alpha \ne 1-n$ is troubling. It is shown in this article that if we normalize by the volume $V(K)$ of the solution K, then there is a solution to the even L_p-Minkowski problem for all $\alpha \le 0$ (with no additional restriction). Note that this *normalized even L_p-Minkowski problem* is equivalent to the even L_p-Minkowski problem for all α except $1-n$.

We first present a solution to the *normalized even discrete L_p-Minkowski problem*:

Theorem 1. *Suppose $\alpha \le 0$. If the unit vectors $u_1, \dots, u_N$ do not lie in a great subsphere of S^{n-1} and $a_1, \dots, a_N > 0$ are given, then there exists a convex polytope P in $\mathbb{R}^n$ that is symmetric about the origin, with $2N$ proper faces, such that if f_i and h_i are the area and support numbers of the faces with outer unit normals $\pm u_i$, then*

$$h_i^\alpha f_i / V(P) = a_i, \qquad \text{for all } i.$$

Furthermore, the polytope is unique if $\alpha < 0$.

This will yield the solution to the normalized L_p-Minkowski problem with even data:

Theorem 2. *Suppose $\alpha \le 0$. If μ is an even Borel measure on S^{n-1} whose support is not contained in a great subsphere of S^{n-1}, then there exists a convex body K, symmetric about the origin, such that*

$$\frac{h(K,\,\cdot\,)^\alpha}{V(K)} dS(K,\,\cdot\,) = d\mu.$$

Furthermore, the body K is unique if $\alpha < 0$.

1. Basics from the Brunn-Minkowski-Firey theory

The Brunn-Minkowski-Firey theory provides the tools for the solution of the L_p-Minkowski problem. For quick reference, the essentials are presented in this section. The Brunn-Minkowski-Firey theory is not a translation-invariant theory. All convex bodies to which this theory is to be applied must have the origin in their interiors. It will be convenient to assume throughout that all convex bodies contain the origin in their interiors, and that p denotes a fixed real number greater than (or equal to) 1.

For convex bodies K, K', and $\lambda, \lambda' \ge 0$ (not both zero), the Minkowski linear combination $\lambda K + \lambda' K'$ is the convex body defined by

$$h(\lambda K + \lambda' K',\,\cdot\,) = \lambda h(K,\,\cdot\,) + \lambda' h(K',\,\cdot\,).$$

For convex bodies K, K' and $\lambda, \lambda' \ge 0$ (not both zero), the Firey L_p-combination $\lambda \cdot K +_p \lambda' \cdot K'$, is defined by

$$h(\lambda \cdot K +_p \lambda' \cdot K',\,\cdot\,)^p = \lambda h(K,\,\cdot\,)^p + \lambda' h(K',\,\cdot\,)^p.$$

Note that "$\cdot$" rather than "$\cdot_p$" is written for Firey scalar multiplication. This should create no confusion. Also note that the relationship between Firey and Minkowski scalar multiplication is $\lambda\cdot K = \lambda^{1/p}K$. Firey L_p-combinations of convex bodies were defined and studied by Firey, who called them p–means of convex bodies (see, e.g., [BZ, pp. 161–162] and [S, pp. 383–384]).

The mixed volume $V_1(K, L)$ of the convex bodies K, L is defined by

$$nV_1(K, L) = \lim_{\varepsilon\to 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon}.$$

For $x \in \mathbb{R}^n$, let $[-x, x]$ denote the convex body that is the closed line segment joining $-x$ to x. From the definition of V_1 it is easily verified that for $u \in S^{n-1}$,

$$nV_1(K, [-u, u]) = 2\,\mathrm{vol}_{n-1}(K|u^\perp),$$

where $\mathrm{vol}_{n-1}(K|u^\perp)$ denotes the area (i.e., $(n-1)$-dimensional volume) of $K|u^\perp$, the orthogonal projection of K onto the codimension-1 subspace of $\mathbb{R}^n$ that is orthogonal to u.

For $p \geq 1$, the L_p-mixed volume $V_p(K, L)$ of the convex bodies K, L was defined in [Lu$_1$] by

$$\frac{n}{p}V_p(K, L) = \lim_{\varepsilon\to 0^+} \frac{V(K +_p \varepsilon\cdot L) - V(K)}{\varepsilon}. \tag{1.1}$$

That this limit exists was demonstrated in [Lu$_1$]. Obviously, for each K,

$$V_p(K, K) = V(K).$$

It was shown by Aleksandrov [A$_1$] and Fenchel & Jessen [FJ] that the mixed volume V_1 has the following integral representation:

$$V_1(K, Q) = \frac{1}{n}\int_{S^{n-1}} h(Q, v)\, dS(K, v),$$

for each convex body Q. Since $nV_1(K, [-u, u]) = 2\,\mathrm{vol}_{n-1}(K|u^\perp)$ for $u \in S^{n-1}$, by taking $Q = [-u, u]$ in the integral representation, we get

$$\frac{1}{2}\int_{S^{n-1}} |v\cdot u|\, dS(K, v) = \mathrm{vol}_{n-1}(K|u^\perp). \tag{1.2}$$

It was shown in [Lu$_1$] that corresponding to each convex body K there is a positive Borel measure $S_p(K, \cdot\,)$ on S^{n-1} such that the L_p-mixed volume V_p has the following integral representation:

$$V_p(K, Q) = \frac{1}{n}\int_{S^{n-1}} h(Q, v)^p\, dS_p(K, v), \tag{1.3}$$

for each convex body Q. It turns out that the L_p-surface area measure $S_p(K, \cdot\,)$ is absolutely continuous with respect to $S(K, \cdot\,)$, and has Radon–Nikodym derivative

$$\frac{dS_p(K, \cdot\,)}{dS(K, \cdot\,)} = h(K, \cdot\,)^{1-p}. \tag{1.4}$$

If P is a polytope with N proper faces with areas $f_1, \dots, f_N$, support numbers $h_1, \dots, h_N$ and corresponding unit normals $u_1, \dots, u_N$, then the measure $S_p(P, \cdot\,)$ is a discrete measure whose support is $\{u_1, \dots, u_N\}$ and such that

$$S_p(P, \{u_i\}) = h_i^{1-p} f_i, \qquad \text{for all } i.$$

The tool used to establish uniqueness in the classical Minkowski problem is the Minkowski mixed volume inequality: For convex bodies K, L in $\mathbb{R}^n$,

$$V_1(K,L)^n \geq V(K)^{n-1}V(L),$$

with equality if and only if K and L are homothets (i.e., there exist $x \in \mathbb{R}^n$ and $\lambda > 0$ such that $K = x + \lambda L$). It was shown in [Lu$_1$] that there is an L_p-Minkowski inequality: If K, L are convex bodies in $\mathbb{R}^n$, and $p > 1$, then

$$V_p(K,L)^n \geq V(K)^{n-p}V(L)^p, \tag{1.5}$$

with equality if and only if K and L are dilates (i.e., there exists a $\lambda > 0$ such that $K = \lambda L$).

The following two facts regarding the L_p-surface area measures are needed in this article. First, if $p > 1$ and $S_p(K, \cdot\,)$ is even, then the convex body K is symmetric about the origin. This fact was established in [Lu$_1$]. The other fact needed is that if a sequence of convex bodies K_i converges, in the Hausdorff topology, to the convex body K, then the sequence of L_p-surface area measures $S_p(K_i, \cdot\,)$ converges weakly to $S_p(K, \cdot\,)$. This can be found in [Lu$_2$, p. 251].

One new but easily established result, from the Brunn-Minkowski-Firey theory, is needed:

Proposition. *If K, L are convex bodies, then*

$$V_p(K,L) = V(K) + \frac{p}{n} \lim_{\lambda \to 1^-} \frac{V(\lambda \cdot K +_p (1-\lambda)\cdot L) - V(K)}{1-\lambda}.$$

Proof. Let

$$l = \lim_{\lambda \to 1^-} \frac{V(\lambda \cdot K +_p (1-\lambda)\cdot L) - V(K)}{1-\lambda}.$$

Since

$$\lambda \cdot K +_p (1-\lambda)\cdot L = \lambda \cdot [K +_p \frac{1-\lambda}{\lambda}\cdot L],$$

we have

$$l = \lim_{\lambda \to 1^-} \frac{\lambda^{n/p} V(K +_p \frac{1-\lambda}{\lambda}\cdot L) - V(K)}{1-\lambda}.$$

Substitute $\varepsilon = (1-\lambda)/\lambda$, and for $\varepsilon \geq 0$ define f, g by $f(\varepsilon) = V(K +_p \varepsilon \cdot L)$ and $g(\varepsilon) = (1+\varepsilon)^{-n/p}$. Hence

$$l = \lim_{\varepsilon \to 0^+} \frac{g(\varepsilon)f(\varepsilon) - g(0)f(0)}{\varepsilon}(1+\varepsilon).$$

But (1.1) gives $f'(0) = \frac{n}{p}V_p(K,L)$, and hence

$$l = \frac{n}{p}[V_p(K,L) - V(K)].$$

□

An immediate consequence of the proposition (that is needed later) is

Corollary. *If K, L are convex bodies, and there exists an $\varepsilon_o < 1$ such that*

$$V(\lambda \cdot K +_p (1-\lambda)\cdot L) \leq V(K), \qquad \textit{for all } \lambda \in (\varepsilon_o, 1),$$

then $V_p(K,L) \leq V(K)$.

2. The L_p-Minkowski problem with even discrete data

A minor reformulation of Theorem 1 of the introduction is:

Theorem 1. *Suppose $p \geq 1$. If $u_1, \dots, u_N$ are N distinct unit vectors that do not lie in a great subsphere of S^{n-1} and $a_1, \dots, a_N > 0$ are given, then there exists a convex polytope P in $\mathbb{R}^n$ that is symmetric about the origin, with $2N$ proper faces, such that if f_i and h_i are the area and support numbers of the two faces with outer unit normals $\pm u_i$, then*

$$h_i^{1-p} f_i / V(P) = a_i, \qquad \text{for all } i.$$

Furthermore, the polytope P is unique. (If $p = 1$, the uniqueness is up to translation.)

An equivalent formulation is in:

Theorem 1′. *Suppose $p \geq 1$. If μ is a discrete even Borel measure whose support is not contained in a great subsphere of S^{n-1}, then there exists a polytope P in $\mathbb{R}^n$ that is symmetric about the origin and such that*

$$S_p(P, \cdot)/V(P) = \mu.$$

Furthermore, the polytope P is unique. (If $p = 1$, the uniqueness is up to translation.)

Let $\mathbb{R}_+^N = \{\mathrm{k} = (\mathrm{k}_1, \dots, \mathrm{k}_N) \in \mathbb{R}^N : \mathrm{k}_i \geq 0 \text{ for all } i\}$. Define the $(N-1)$-dimensional surface M by

$$M = \{\mathrm{k} \in \mathbb{R}_+^N : \frac{1}{n} \sum_{i=1}^N a_i \, \mathrm{k}_i^p = 1\}.$$

Since all the $a_i > 0$, the surface M is compact. For each $\mathrm{k} \in M$, define the compact convex set $\mathbf{k}$ by

$$\mathbf{k} = \{x \in \mathbb{R}^n : |x \cdot u_i| \leq \mathrm{k}_i \ \text{ for all } i\}.$$

The polytope $\mathbf{k}$ is symmetric about the origin and has at most $2N$ proper faces whose outer unit normals are from the set $\{\pm u_1, \dots, \pm u_N\}$. The important fact here is that, in general,

$$h(\mathbf{k}, \pm u_i) \leq \mathrm{k}_i;$$

however, if $\mathbf{k}$ has a proper face (i.e. with non-zero area) orthogonal to u_i, then in fact

$$h(\mathbf{k}, \pm u_i) = \mathrm{k}_i\,.$$

Since M is compact and the function $\mathrm{k} \mapsto V(\mathbf{k})$ is continuous, there exists a point $\bar{\mathrm{k}} \in M$ such that

$$V(\mathbf{k}) \leq V(\bar{\mathbf{k}}) \qquad \text{for all } \mathrm{k} \in M.$$

Note that the minimum of the function $\mathrm{k} \mapsto V(\mathbf{k})$ occurs on the boundary of the surface M: For each point k on the boundary of the surface M we have some $\mathrm{k}_i = 0$, and hence $V(\mathbf{k}) = 0$.

We first show that

$$V(\bar{\mathbf{k}}) \geq V_p(\bar{\mathbf{k}}, \mathbf{k}), \qquad \text{for all } \mathrm{k} \in M. \tag{2.1}$$

To do so, suppose $\mathrm{k} \in M$ and $\lambda \in [0,1]$. Define $\hat{\mathrm{k}} \in \mathbb{R}_+^N$ by

$$\hat{\mathrm{k}}_i = (\lambda \, \bar{\mathrm{k}}_i^p + (1-\lambda) \, \mathrm{k}_i^p)^{1/p}.$$

Now, since $\bar{\mathrm{k}}, \mathrm{k} \in M$,

$$\frac{1}{n}\sum_{i=1}^{N} a_i\, \bar{\mathrm{k}}_i^p = 1 = \frac{1}{n}\sum_{i=1}^{N} a_i\, \mathrm{k}_i^p,$$

and hence $\frac{1}{n}\sum_{i=1}^{N} a_i\, \hat{\mathrm{k}}_i^p = 1$, which shows that $\hat{\mathrm{k}} \in M$. Now

$$h(\lambda\cdot\bar{\mathbf{k}} +_p (1-\lambda)\cdot\mathbf{k}, \pm u_i)^p = \lambda h(\bar{\mathbf{k}}, \pm u_i)^p + (1-\lambda) h(\mathbf{k}, \pm u_i)^p \le \lambda\, \bar{\mathrm{k}}_i^p + (1-\lambda)\, \mathrm{k}_i^p = \hat{\mathrm{k}}_i^p .$$

This shows that

$$\lambda\cdot\bar{\mathbf{k}} +_p (1-\lambda)\cdot\mathbf{k} \subset \hat{\mathbf{k}},$$

and from the maximality of $V(\bar{\mathbf{k}})$ we have

$$V(\bar{\mathbf{k}}) \ge V(\hat{\mathbf{k}}) \ge V(\lambda\cdot\bar{\mathbf{k}} +_p (1-\lambda)\cdot\mathbf{k}).$$

The desired result (2.1) now follows immediately from the previously established corollary.

Let $\bar{f}_1, \dots, \bar{f}_N$ denote the areas and $\bar{h}_1, \dots, \bar{h}_N$ denote the support numbers of the faces of $\bar{\mathbf{k}}$ whose outer unit normals are $\pm u_1, \dots, \pm u_N$. While it can be easily seen that since $\bar{\mathbf{k}}$ has maximal volume, $\bar{h}_i = \bar{\mathrm{k}}_i$, for all i, this will follow from other considerations at the end of the proof. For now, the only fact that is to be used is that while in general $\bar{h}_i \le \bar{\mathrm{k}}_i$, if however $\bar{f}_{i_o} > 0$, then $\bar{h}_{i_o} = \bar{\mathrm{k}}_{i_o}$. Define

$$\bar{a}_i = \bar{f}_i\, \bar{\mathrm{k}}_i^{1-p}, \qquad \text{for } i = 1, \dots, N.$$

There exists a neighborhood $U = U(\bar{\mathrm{k}})$ of $\bar{\mathrm{k}}$ in M with the following property: If $\bar{\mathbf{k}}$ has a proper face (i.e., with positive area) orthogonal to a direction u_{i_o}, then for each $\mathrm{k} \in U$, the polytope $\mathbf{k}$ has this property (i.e., has a proper face orthogonal to the direction u_{i_o}). Hence, if for a particular i we have $\bar{f}_i > 0$, then $h(\mathbf{k}, u_i) = \mathrm{k}_i$ for all $\mathrm{k} \in U$. Thus, for all $\mathrm{k} \in U$,

$$V_p(\bar{\mathbf{k}}, \mathbf{k}) = \frac{1}{n}\sum_{i=1}^{N} \bar{f}_i\, \bar{\mathrm{k}}_i^{1-p}\, h(\mathbf{k}, u_i)^p = \frac{1}{n}\sum_{i=1}^{N} \bar{f}_i\, \bar{\mathrm{k}}_i^{1-p}\, \mathrm{k}_i^p = \frac{1}{n}\sum_{i=1}^{N} \bar{a}_i\, \mathrm{k}_i^p . \tag{2.2}$$

In particular, choosing $\bar{\mathrm{k}}$ for k gives

$$V(\bar{\mathbf{k}}) = V_p(\bar{\mathbf{k}}, \bar{\mathbf{k}}) = \frac{1}{n}\sum_{i=1}^{N} \bar{a}_i\, \bar{\mathrm{k}}_i^p . \tag{2.3}$$

Define the surface

$$\tilde{M} = \{\mathrm{k} \in \mathbb{R}_+^N : \frac{1}{n}\sum_{i=1}^{N} \bar{a}_i\, \mathrm{k}_i^p = V(\bar{\mathbf{k}})\}.$$

From (2.3) it follows immediately that $\bar{\mathrm{k}} \in \tilde{M}$. Hence, the surfaces U and $\tilde{M}$ have $\bar{\mathrm{k}}$ as a common point.

By combining (2.1) and (2.2) we see that for all $\mathrm{k} \in U$,

$$\frac{1}{n}\sum_{i=1}^{N} \bar{a}_i\, \mathrm{k}_i^p \le V(\bar{\mathbf{k}}).$$

This and the definition of $\tilde{M}$ show that the surface $\tilde{M}$ is tangent to the surface U at the point $\bar{\mathrm{k}} \in U \cap \tilde{M}$. Taking gradients of U and $\tilde{M}$ at the point $\bar{\mathrm{k}}$ shows the existence of a $c > 0$ such that

$$\tfrac{p}{n}(a_1\, \bar{\mathrm{k}}_1^{1-p}, \dots, a_N\, \bar{\mathrm{k}}_N^{1-p}) = c\tfrac{p}{n}(\bar{a}_1\, \bar{\mathrm{k}}_1^{1-p}, \dots, \bar{a}_N\, \bar{\mathrm{k}}_N^{1-p}).$$

Since $V(\bar{\mathbf{k}}) > 0$, all the $\bar{\mathrm{k}}_i > 0$. Hence

$$a_i = c\bar{a}_i \qquad \text{for all } i.$$

Now $\bar{\mathrm{k}} \in U$ gives $\frac{1}{n}\sum_{i=1}^N a_i\, \bar{\mathrm{k}}_i^p = 1$, which in turn now gives $c\frac{1}{n}\sum_{i=1}^N \bar{a}_i\, \bar{\mathrm{k}}_i^p = 1$. But $\bar{\mathrm{k}} \in \tilde{M}$ gives $\frac{1}{n}\sum_{i=1}^N \bar{a}_i\, \bar{\mathrm{k}}_i^p = V(\bar{\mathbf{k}})$, and hence $c = 1/V(\bar{\mathbf{k}})$. Hence, $\bar{a}_i = V(\bar{\mathbf{k}})a_i$ for all i, or equivalently

$$\bar{f}_i\, \bar{\mathrm{k}}_i^{1-p} = V(\bar{\mathbf{k}})a_i, \qquad \text{for all } i.$$

Since $a_i > 0$ for all i, this shows that $\bar{f}_i > 0$ for all i, which in turn gives $\bar{h}_i = \bar{\mathrm{k}}_i$ for all i. Hence

$$\bar{f}_i\bar{h}_i^{1-p} = V(\bar{\mathbf{k}})a_i, \qquad \text{for all } i,$$

which completes the existence part of the proof.

To see that the solution is unique, suppose that there are two solutions, say P and P'. Hence,

$$S_p(P,\,\cdot\,)/V(P) = S_p(P',\,\cdot\,)/V(P').$$

From this and the integral representation (1.3) we conclude that for all convex bodies Q,

$$\frac{V_p(P,Q)}{V(P)} = \frac{V_p(P',Q)}{V(P')}.$$

Now take P' for Q, Use the L_p-Minkowski inequality (1.5) and the fact that $V_p(P',P') = V(P')$, to get $V(P) \geq V(P')$ with equality if and only if P and P' are dilates. (For $p = 1$, with equality if and only if P and P' are homothets.) By choosing P for Q, we see similarly that in fact $V(P) = V(P')$, and hence from the equality conditions we see that P and P' are identical (for $p = 1$, identical up to translation).

3. The L_p-Minkowski problem with even data

To prove that the solution of the L_p-Minkowski problem with even data follows from the solution of the L_p-Minkowski problem with even discrete data involves fairly standard approximation arguments. However, for the L_p-Minkowski problem new a priori estimates are required to show that the minimizing sequence is bounded from below as well as from above.

Theorem 2. *Suppose $p \geq 1$. If μ is an even Borel measure on S^{n-1} whose support is not contained in a great subsphere of S^{n-1}, then there exists a convex body K, symmetric about the origin, such that*

$$\frac{h(K,\,\cdot\,)^{1-p}}{V(K)}dS(K,\,\cdot\,) = d\mu.$$

Furthermore, the body K is unique. (If $p = 1$, the body is unique up to translation.)

For each positive integer i, partition S^{n-1} into a finite collection $\mathcal{P}_i$ of Borel sets, such that for each $\Delta \in \mathcal{P}_i$ its antipodal set $-\Delta$ is also in $\mathcal{P}_i$, and $\mathrm{diam}(\Delta) < 1/i$ for each $\Delta \in \mathcal{P}_i$. For each $\Delta \in \mathcal{P}_i$ choose $c_\Delta \in \Delta$ so that $c_{-\Delta} = -c_\Delta$, and define the Borel measure μ_i on S^{n-1} by letting

$$\int_{S^{n-1}} f\, d\mu_i = \sum_{\Delta \in \mathcal{P}_i} f(c_\Delta)\mu(\Delta),$$

for each measurable f. Obviously, each μ_i is an even discrete measure, and it is easily seen that the sequence of measures μ_i converges weakly to μ.

For each even Borel measure ϕ on S^{n-1}, consider the function defined on $\mathbb{R}^n$ by

$$x \longmapsto \frac{1}{n}\int_{S^{n-1}} |x\cdot v|^p d\phi(v).$$

From the Minkowski integral inequality it follows that the p-th root of this function is convex and hence is the support function of a convex body. Let $\Pi_p\phi$ denote this body; i.e., define $\Pi_p\phi$ by

$$h(\Pi_p\phi, u)^p = \frac{1}{n}\int_{S^{n-1}} |u\cdot v|^p d\phi(v),$$

for $u \in S^{n-1}$. Obviously, the support of an even measure ϕ is not contained in a great subsphere of S^{n-1} if and only if the continuous function $h(\Pi_p\phi,\,\cdot\,)$ is strictly positive on S^{n-1}, or equivalently if and only if the body $\Pi_p\phi$ contains the origin in its interior.

Since the support of μ does not lie on a great subsphere of S^{n-1}, the convex body $\Pi_p\mu$ contains the origin in its interior. Hence there exist $a, b > 0$ such that $a/2 \geq h(\Pi_p\mu,\,\cdot\,) \geq 2b$ on S^{n-1}. Since $\mu_i \to \mu$ weakly, it follows that $h(\Pi_p\mu_i,\,\cdot\,) \longrightarrow h(\Pi_p\mu,\,\cdot\,)$ pointwise on S^{n-1}. But the pointwise convergence of support functions is, in fact, a uniform convergence on S^{n-1} (see, e.g., Schneider [S, p. 54]). Hence, there exists an integer i_o such that on S^{n-1},

$$a \geq h(\Pi_p\mu_i,\,\cdot\,) \geq b > 0, \qquad \text{for all } i \geq i_o.$$

This shows (among other things) that for all $i \geq i_o$ the supports of the measures μ_i do not lie in a great subsphere of S^{n-1}.

For each $i \geq i_o$, we now use Theorem 1′ to get a polytope P_i, symmetric about the origin, such that

$$S_p(P_i,\,\cdot\,)/V(P_i) = \mu_i. \tag{3.1}$$

To see that the diameters of the polytopes P_i are bounded, define real M_i and some $u_i \in S^{n-1}$ by

$$M_i = \max_{u\in S^{n-1}} h(P_i, u) = h(P_i, u_i).$$

Now, $M_i[u_i, -u_i] \subset P_i$, where as before $[u_i, -u_i]$ denotes the closed line segment joining u_i and $-u_i$. Hence, $M_i|u_i\cdot v| \leq h(P_i, v)$ for all $v \in S^{n-1}$. Thus, for all $i \geq i_o$

$$M_i^p b^p \leq M_i^p \frac{1}{n}\int_{S^{n-1}} |u_i\cdot v|^p d\mu_i(v) \leq \frac{1}{n}\int_{S^{n-1}} h(P_i, v)^p \frac{dS_p(P_i, v)}{V(P_i)} = \frac{V_p(P_i, P_i)}{V(P_i)} = 1.$$

Thus, $M_i \leq 1/b$ for sufficiently large i, and hence the sequence of bodies $\{P_i\}$ is bounded from above.

For the L_p-Minkowski problem it is critical to show that the sequence $\{P_i\}$ is bounded from below as well as from above. To this end, define real m_i and a $v_i \in S^{n-1}$ by

$$m_i = \min_{u\in S^{n-1}} h(P_i, u) = h(P_i, v_i).$$

Since each P_i contains the origin in its interior, each $m_i > 0$. The fact that $a \geq h(\Pi_p\mu_i,\,\cdot\,)$, for $i \geq i_o$, together with (3.1), (1.4), Jensen's inequality, and

(1.2), shows that, for $i \geq i_o$,

$$\begin{aligned} a &\geq \left(\frac{1}{n} \int_{S^{n-1}} |v_i \cdot u|^p d\mu_i(u) \right)^{\frac{1}{p}} = \left(\frac{1}{n} \int_{S^{n-1}} |v_i \cdot u|^p \frac{dS_p(P_i, u)}{V(P_i)} \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{n} \int_{S^{n-1}} \left(\frac{|v_i \cdot u|}{h(P_i, u)} \right)^p \frac{h(P_i, u) dS(P_i, u)}{V(P_i)} \right)^{\frac{1}{p}} \\ &\geq \frac{1}{n} \int_{S^{n-1}} |v_i \cdot u| \, \frac{dS(P_i, u)}{V(P_i)} = \frac{2}{nV(P_i)} \operatorname{vol}_{n-1}(P_i | v_i^{\perp}). \end{aligned}$$

Since P_i is contained in the right cylinder $(P_i|v_i^{\perp}) \times [-h(P_i, v_i)v_i, h(P_i, v_i)v_i]$, we have

$$2m_i \operatorname{vol}_{n-1}(P_i|v_i^{\perp}) = 2h(P_i, v_i) \operatorname{vol}_{n-1}(P_i|v_i^{\perp}) \geq V(P_i).$$

Thus,

$$a \geq \frac{2}{n} \frac{\operatorname{vol}_{n-1}(P_i|v_i^{\perp})}{V(P_i)} \geq \frac{1}{nm_i},$$

which shows that $m_i \geq \frac{1}{na}$, for sufficiently large i.

Since the sequence of bodies $\{P_i\}$ is bounded from above, by the Blaschke selection theorem there exists a subsequence, which we also denote by $\{P_i\}$, which converges to a convex body, say K. Since the P_i are symmetric about the origin, the body K is symmetric about the origin as well. Since $m_i \geq 1/na$ for sufficiently large i, we know that K contains the origin in its interior. Since $P_i \longrightarrow K$ and K contains the origin in its interior, the L_p surface area measures $S_p(P_i, \cdot)$ converge weakly to $S_p(K, \cdot)$, and $1/V(P_i)$ converges to $1/V(K)$. Thus the measures

$$\frac{S_p(P_i, \cdot)}{V(P_i)} \longrightarrow \frac{S_p(K, \cdot)}{V(K)} \qquad \text{weakly on } S^{n-1}.$$

But from (3.1), $S_p(P_i, \cdot)/V(P_i) = \mu_i$, and the μ_i converge weakly to μ. Hence,

$$\frac{S_p(K, \cdot)}{V(K)} = \mu.$$

The uniqueness part of Theorem 2 follows in exactly the same manner as the uniqueness part of Theorem 1.

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Department of Mathematics, Polytechnic University, Brooklyn, New York 11201
E-mail address: elutwak@poly.edu

Department of Mathematics, Polytechnic University, Brooklyn, New York 11201
E-mail address: dyang@poly.edu

Department of Mathematics, Polytechnic University, Brooklyn, New York 11201
E-mail address: gzhang@poly.edu